AN ELEMENTARY TREATISE ON MECHANICS;

DESIGNED AS A TEXT-BOOK FOR THE UNIVERSITY EXAMINATIONS
FOR THE ORDINARY DEGREE OF B. A.

PART I.

STATICS.

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PREFACE TO FIRST EDITION.

This Treatise contains the text of the Lectures which I have delivered for some years past in University College.

As my design has been only to furnish to Students a text-book for such parts of the subject as are required for the ordinary degree of B. A. in Universities, I have not thought it advisable to burden this work with mere explanation or illustration, or to add examples; presuming that such, where necessary, will be furnished in the Lecture-room or by the Tutor.

UNIVERSITY COLLEGE,
TORONTO,
April 1, 1858.

CHAPTER I.

DEFINITIONS AND PRINCIPLES.

1. A *material particle* is a portion of matter occupying an indefinitely small space; or, a geometrical point endowed with the properties of matter. Material particle.

All bodies may be geometrically conceived as made up of particles.

2. When the distance between two particles remains unchanged during any period of time, they are relatively at rest, and we conceive that they will continue so unless one or both be acted on by some cause to which we give the name of *Force.* Force.

The state of rest or motion of a particle can only be conceived of in relation to others, but it is convenient to speak of it *absolutely* as being at rest or in motion, reference being understood to ourselves (or some particles in a known relation to ourselves), and changes of rest or motion are to be considered as produced by forces acting on the particle alone.

3. When a particle at rest is set in motion by a force, it will begin to move in a particular line, which we may define to be the *direction* or *line of action* of the force. The motion might be just prevented and the particle kept at rest, by a suitable force applied in an opposite direction. In this case the two forces are said to balance or counter-balance each other; and the magnitudes of two forces are said to be equal when each would separately counter-balance the same force. Direction and magnitude of a Force. Equal Forces.

4. Generally, when forces acting on any system of particles keep them at rest, the forces are said to counter-balance, or Statics, general problem of.

to be in equilibrium; and the investigation of the relations among them in such case, or the conditions of equilibrium, constitutes the science of *Statics*.

Force, how measured.

5. Some convenient force being assumed as a standard or *unit*, the magnitude of any force is measured numerically with reference to the unit by the number of such units (acting simultaneously at a point and opposite to the force), which it will counter-balance. Thus, if the force will counter-balance n unit forces, its magnitude is said to be n.

This supposes n to be a whole number, and we can always take a unit-force of such magnitude that it shall be so; then, when any other force is taken as the unit, the magnitude of our original force will be expressed by the ratio (whether a whole number or a fraction) which its magnitude bears to that of the force assumed as unit.

In general, the term *Pressure* may be used for a Force thus statically considered and measured.

Weight.

6. It is found that on all bodies on the Earth a pressure is exerted downwards, in a vertical direction: that is, in a direction perpendicular to the surface of still water at the place; and this pressure (which, for any particular body, is called the *weight* of that body) is invariable at the same place for the same body at all times, whatever form, size or position the body be made to take. Hence the *weight* of some particular body may conveniently be assumed as a unit to which other pressures may be referred for measurement.

Unit of Force.

In the English system the weight of a certain piece of brass carefully preserved as a standard, is called one pound Troy, and all other weights are referred to this. If lost, it might be restored from the knowledge that this *pound* being divided into 5760 *grains*, "a cubic inch of distilled water, of the temperature of 62° Fahrenheit, when the barometer is at 30 inches, weighs 252.724 grains"

Rigidity.

7. When a system of particles, or a body, is such that the relative distances of the particles undergo no change by the action of the forces applied to them in any manner, the system is said to be *rigidly* connected, and the body is called a *rigid* body, relatively to the forces concerned.

8. From the definitions laid down, it will be observed that three elements enter into every Force: (1), its point of application, or the particle on which it acts; (2), its direction; (3), its magnitude. When these are known, the force is fully determined.

9. The following is the fundamental law, deduced from experiment, on which the Science of Statics is based: Experimental Law.

If two equal forces act respectively on two particles, which are rigidly connected, in the line joining them but in opposite directions, they will counterbalance.

Hence, either of these forces may be transferred to the other particle, preserving the same direction, without alteration of its statical effect; or:

"*A force may be supposed to act at* ANY *point in its own line of action, the new point of application being rigidly connected with the former one;* and in this latter form the law is frequently stated.

10. The following consequences may be noted:

When a pressure is communicated by means of a straight rigid rod in direction of its length, the pressure is wholly effective in this direction and may be supposed to act at any point of the rod.

When an inextensible string is stretched straight by a force at each end, these two forces must be equal, and their magnitude is independent of the length of the string. Also at *every* point of the string there are acting along it, in opposite directions, two forces equal to the former, and either of these is called the *tension* of the string, which is thus uniform throughout its whole length. The same is true when the string (if it be perfectly flexible) is stretched over a *smooth* surface. Tension of a String.

A *smooth* surface is one which can exert a pressure at any point of it, only in direction of the normal at that point.

Forces can be represented by straight lines.

11. Since the three elements which serve to determine a pressure are in their nature identical with those which determine a straight line—namely, magnitude, direction, and point through which it is to be drawn—it follows that a straight line may properly be taken as the representative of a pressure. When, however, a line $A B$ is so taken, it will be understood that the pressure acts in the direction from A towards B; if written $B A$, then from B towards A. Frequently also, the words "represented by" will be omitted, and we shall use "the force $A B$" to indicate the force represented in magnitude and direction by the line $A B$, acting in a direction from A towards B.

12. We now proceed to state the two problems of Statics which alone will be here touched upon.

(1). The conditions of equilibrium for any set of Forces acting on the same particle.

(2). The conditions of equilibrium when Forces act on a rigid system of particles which has a fixed axis round which it can turn freely, the Forces acting perpendicularly to this axis.

CHAPTER II.

FORCES ACTING AT A POINT.

Definition of Resultant.

13. When Forces act simultaneously on a particle at rest, if the particle begin to move, its motion will commence in a definite direction, and might be just prevented by a single force of suitable magnitude applied in an opposite direction. This force would then counterbalance the original set of Forces, and a force equal and opposite to it would produce the same statical effect as the first set of Forces, and is therefore termed their *Resultant.*

14. Hence, when any set of Forces acting at a point keep it at rest, since any one of them may be considered as counterbalancing all the rest, a force equal and opposite to any one of them is the Resultant of all the others.

Condition of Equilibrium

15. Hence also the condition, in order that Forces acting at a point may keep it at rest, is that the magnitude of their Resultant shall be zero, or that their Resultant shall vanish.

Forces in the same line.

16. When Forces act in the same line and direction on a point, their Resultant acts in the same direction, and its magnitude is equal to the sum of their magnitudes. If some of the Forces be acting in the opposite direction, the magnitude of their Resultant will be the difference between the sums of the magnitude of those acting in the one direction and in the other, and it will act in the same direction as those Forces whose sum is the greater. We can, however, indicate oppositeness of direction by attaching to the magnitudes of the Forces the Algebraic signs $+$ and $-$; so that, any one Force being considered *positive*, all Forces in that direction

will also be considered *positive*, but forces in the opposite direction will be considered *negative*.

The above results may then be combined into the following:

Their Resultant and

The Resultant of any set of Forces acting on a point in the same line, is the algebraic sum of the Forces.

Condition of Equilibrium

17. Hence also the condition that the point may be kept at rest will be that

The algebraic sum of the Forces shall be zero.

Equal oblique Forces.

18. If two equal forces act in different directions at a point, their Resultant will act in a direction bisecting the angle between their directions.

Any two Forces.

19. *The Principle of the* PARALLELOGRAM OF FORCES.

Parallelogram of Forces.

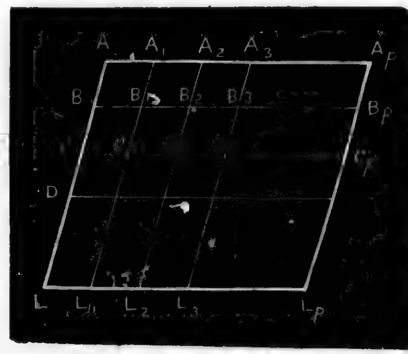
Newton.

If two Forces acting on a particle be represented in magnitude and direction by two straight lines drawn from a point, and the parallelogram, of which these lines are adjacent sides, be completed, that diagonal which passes through the point will represent in magnitude and direction the Resultant of the two forces.

Direction of Resultant.

Duhamel's proof.

Let the lines AA_p, AL be drawn representing in magnitude and direction two forces acting at A; and let p, q be the numbers denoting the magnitudes of the forces.



Divide AA_p into p equal parts in the points A_1, A_2, A_3, ..., and AL into q equal parts in the points B, C, D, ...: then each of these equal parts will represent in magnitude the unit force. Through these points, draw lines parallel to

the original lines, completing the parallelogram; and suppose all the lettered points of the figure rigidly connected.

Then, since the two forces represented by AA_1, AB, acting at A, are equal, the direction of their resultant bisects the angle between them, and it therefore acts in AB_1: it may then be supposed to act at B_1 (§ 9), and may there be again resolved into its original components, which will be represented by BB_1 and A_1B_1, of which the former may act at B, and the latter at A_1.

Proceeding in the same way with this latter force, A_1B_1, at A_1, and the force A_1A_2, which we may also take to act at A_1, we can replace these by B_1B_2 at B_1, and A_2B_2 at A_2.

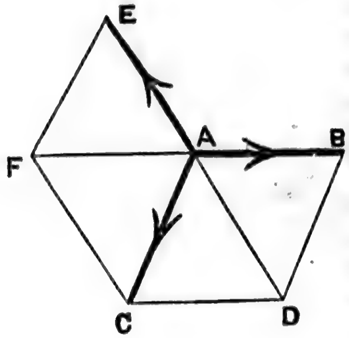
Proceeding in this manner we arrive at last at B_p, where we find the force A_pB_p and the set of forces BB_1, B_1B_2, B_2B_3, ... (which latter make up the original force p) as the equivalents of p and AB at A.

Now taking up the set of forces in BB_p and the force represented by BC at B, we transform them by the same process into B_pC_p and the set in CC_p at C_p.

Following this method we arrive at last at L_p, where we have for the equivalents of the original forces the sets of forces in LL_p and A_pL_p, which may be supposed all to act at L_p and their magnitudes are p and q. Hence we have transformed the original forces p and q acting at A to the same forces acting at L_p in parallel directions to the former, and this without alteration of their statical effect. Hence L_p must be a point in the direction of the Resultant of the original forces at A; that is AL_p is the direction of the Resultant, which proves the principle enunciated, so far as the direction of the Resultant is concerned.

Magnitude of Resultant.

Let AB, AC, represent the two forces acting at A. Complete the parallelogram $AC\ DB$. Then AD is the direction in which the Resultant acts, and we have now to prove that AD represents also its magnitude.



In DA produced backwards take AE to represent this magnitude, so that a force represented by AE will be equal and opposite to the Resultant; and the three forces represented by AB, AC, AE, will keep the point A at rest, and each one of them is equal and opposite to the Resultant of the other two.

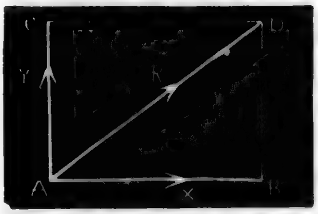
Complete the parallelogram $AEFC$; then, AF is the direction of the Resultant of the forces AE, AC, and is therefore opposite to AB. Hence, FAB is a straight line, and therefore $FACD$ a parallelogram. Hence, the lines AE and AD are equal, being each equal to FC: but AE was taken to represent in magnitude the Resultant of AB, AC, and consequently AD also represents it in magnitude. *Q. E. D.*

Resolution of a Force.

20. Conversely, a force acting at a point can be resolved into an equivalent pair of forces at that point in an infinite number of ways; for, taking a line drawn from a point to represent the Force, and constructing on it as diagonal *any* parallelogram, the two adjacent sides terminating at this point, will represent an equivalent pair of forces.

If this pair consist of two forces acting in perpendicular directions, each of thom is called the *effective* part of the original force in this direction.

Thus, if R be the force at A, represented by AD, and it be resolved into two forces in perpendicular directions—namely, X along AB and Y along AC; then X and Y are the effective parts of R resolved along AB and AC respectively. Completing the rectangle $A\ C\ D\ B$, X and Y will be respectively represented by AB, AC, and, calling the angle BAD, θ, we have from the right-angled triangle $B\ A\ D$,



$$X = R \cos \theta,\ Y = R \sin \theta.$$

21. Hence, to find the effective part of a force in any given direction, or, more briefly, to resolve a force in any given direction, *multiply its magnitude by the cosine of the angle contained between its direction and the given direction:* and to resolve a force perpendicularly to a given direction, *multiply it by the sine of this angle.* Rule for.

22. So also from the same figure we obtain the Resultant (R) of two perpendicular forces (X, Y), and the angle (θ) which its direction makes with one of them (X); for

$$R^2 = X^2 + Y^2;\ \text{and, } \tan\theta = \frac{Y}{X}.$$

23. When any number of Forces act at a point, their whole effect in any direction will be the Algebraic sum of the separate resolved Forces in this direction, which will evidently therefore be equal to their Resultant resolved in the same direction.

Hence also the algebraic sum of the separate resolved forces in direction of the Resultant is the Resultant itself, and the corresponding sum in a direction perpendicular to the Resultant is zero.

24. *To find the magnitude and direction of the Resultant of any forces acting at a point, their directions being all in one plane.* Resultant of any forces in one plane, and.

Taking any two directions at right angles to each other, let each force be resolved into its components in these directions. Let the algebraic sum of these resolved parts in the one direction be X, and in the other Y.

Then the whole set of Forces is equivalent to the two X, Y.

Hence, if R be the Resultant of the whole set, and therefore also of the two X, Y, and θ the angle it makes with the direction of X, the equations in § 22 give

$$R^2 = X^2 + Y^2, \tan \theta = \frac{Y}{X},$$

which determine the Resultant in magnitude and direction. We have also the equivalent relations

$$X = R \cos \theta, \ Y = R \sin \theta.$$

Conditions of equilibrium.

25. *To find the conditions of equilibrium when any Forces act at a point, their directions being in one plane.*

Retaining the notation and method of the last article, since the only condition, in order that the point acted on by the Forces may be kept at rest, is (§ 15) that the Resultant of the Forces must be zero; that is, $R = 0$, we have

$$X = 0, \ Y = 0.$$

And, conversely, if $X = 0$, and $Y = 0$, then we also have $R = 0$, and the point will be kept at rest; hence the necessary and sufficient conditions of equilibrium are that

The algebraic sums of the Forces resolved into two perpendicular directions shall separately vanish.

This principle will be cited under the name of "The vanishing of the Resultant."

26. The process might be readily extended to forces not all acting in one plane.

Thus, if three forces not in one plane act at a point, and three lines be drawn representing them in magnitude and direction; then, if the parallelopiped, of which these lines are adjacent edges, be completed,

that diagonal of it which passes through the point will represent in magnitude and direction the Resultant of the Forces.

Also, if any number of Forces act at a point, the necessary and sufficient conditions of equilibrium are that the algebraic sums of the Forces resolved along three mutually perpendicular directions shall separately vanish.

The following propositions are historically interesting, though included in what has preceded.

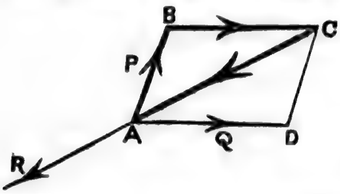
27. *Triangle of Forces.*

Triangle of forces.

If the directions of three forces acting at a point, be parallel to the sides of a triangle taken in order, and their magnitudes be proportional to these sides, they will keep the point at rest.

Stevinus.

For if ABC be the triangle, and A the point at which the forces act; then, completing the parallelogram $ABCD$, the two forces represented by AB, BC, will be represented by $A\,B$, $A\,D$, and their resultant by $A\,C$, which is equal and opposite to CA, the third force.



28. Conversely. If three forces acting at a point and keeping it at rest, be represented in direction by the sides of a triangle taken in order, these sides will represent them also in magnitude.

29. Hence, all problems relative to three forces keeping a point at rest are reduced to the solution of a plane triangle. Thus, if P, Q, R, be the forces, and the angle between P and Q be represented by (P, Q); then the angles of the triangle in the above proposition are the supplements of the angles between the forces; and, since the sine of an angle is equal to that of its supplement, and the cosine of an angle is the cosine of the supplement with opposite sign, we have (Trig. §34, 40.

$$\frac{P}{\text{Sin }(Q,R)} = \frac{Q}{\text{Sin }(R,P)} = \frac{R}{\text{Sin }(P,Q)},$$

Lami's Formulas.

and also the equivalent expressions—

$$R^2 = P^2 + Q^2 + 2\,P\,Q \cos(P, Q)$$
$$P^2 = Q^2 + R^2 + 2\,Q\,R \cos(Q, R)$$
$$Q^2 = R^2 + P^2 + 2\,R\,P \cos(R, P)$$

Polygon of Forces.

30. Polygon of Forces.

If Forces acting on a point be represented in magnitude and direction by the sides of a polygon, taken in order, they will keep the point at rest.

For if $ABCDEF$ be the polygon, the forces AB, BC, have for their resultant AC; and the resultant of this and CD is AD; and so on till we come to the last side which is equal and opposite to the resultant of all the previous ones.

Hence the proposition, as well as its converse, is established.

31. In this way, the Resultant of any number of Forces at a point can be constructed geometrically; for, having drawn consecutive lines, so that, taken in order, they are parallel to, in the same direction with, and proportional in magnitude to, the forces; the line drawn to complete the polygon will represent in magnitude and in reversed direction the Resultant required.

It may be noticed that the Polygon referred to need not be a *plane* one, neither are re-entering angles or crossed sides excluded.

CHAPTER III.

FORCES IN ONE PLANE ACTING ON A SYSTEM OF RIGIDLY CONNECTED POINTS, WHICH CAN TURN FREELY ABOUT A FIXED POINT IN THE PLANE.

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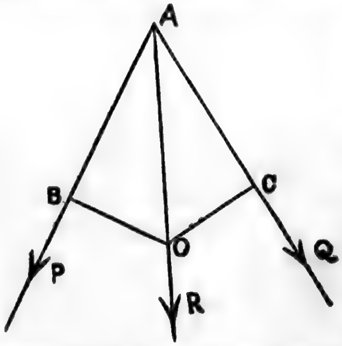
32. *Two intersecting forces act on a rigid system, in the same plane with a fixed point round which the system can turn.*

Condition o equilibrium when the two forces meet.

Let O be the fixed point; P, Q, two forces in the same plane with O, their directions intersecting in A, at which point, rigidly connected with O, they may be supposed to act.



Then if R be the Resultant of P, Q, in order that the point A and the whole system with which it is rigidly connected may be kept at rest, it is necessary and sufficient that the direction of R shall pass through the fixed point O: that is, AO must be the direction of R. Draw OB, OC perpendicular to the directions of P, Q. Then, resolving the forces at A in a direction perpendicular to AO, we have (§ 21, 23):

$$P \sin OAB - Q \sin OAC = 0, \text{ and therefore}$$

$$P.\,OB - Q.\,OC = 0, \text{ or,}$$

$$P.\,OB = Q.\,OC.$$

Moment defined.

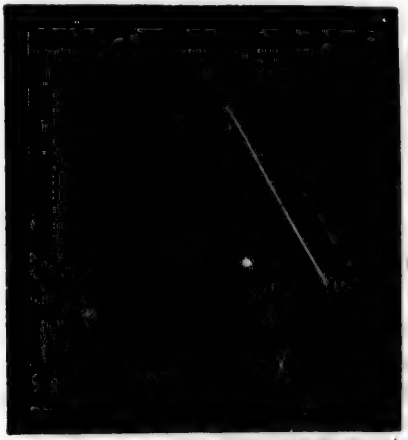
33. The product $P.\ OB$, which is the product of the number expressing the magnitude of the force, and the length of the perpendicular dropped from the point O upon its direction, is called *the moment of the Force about that point.*

Hence the above result may be expressed by saying that when the two forces keep the system at rest, *their moments about the fixed point are equal*, the forces tending to turn the system in opposite directions about the point: but if we indicate this oppositeness of direction by difference of algebraic sign, so that the moment of one Force which tends to turn the system round in *one* direction being considered *positive*, that of another Force tending to turn it in the opposite direction will be considered *negative*; we may still more briefly express it in the form:

The algebraic sum of the moments of the two Forces round the point must be zero.

Moment of resultant of two forces which intersect is equal to sum of moments of forces.

34. *The moment of the Resultant of two intersecting Forces round any point in their plane is equal to the algebraic sum of the moments of the Forces.*



Let P, Q be the two forces intersecting in A; O, any point in their plane; R, their Resultant.

Draw the perpendiculars OB, OC, OD.

Then (§ 23) the Resultant resolved in any direction being equal to the algebraic sum of the resolved Forces in that direction, let them be resolved perpendicularly to OA. Hence, from § 21,

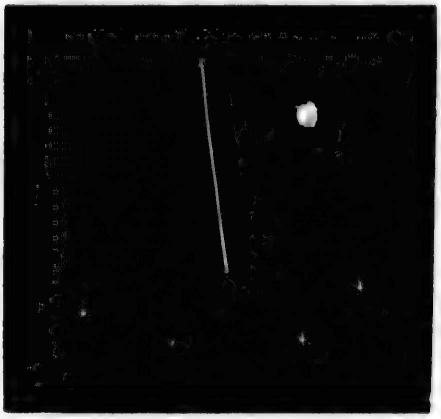
$$R \sin OAD = P \sin OAB + Q \sin OAC.;$$

and therefore

$$R.\ OD = P.\ OB + Q.\ OC,$$

which proves the proposition.

Cor. We have taken the case in which the moments of the Forces have the same sign, the proof in this case being sufficient for all.



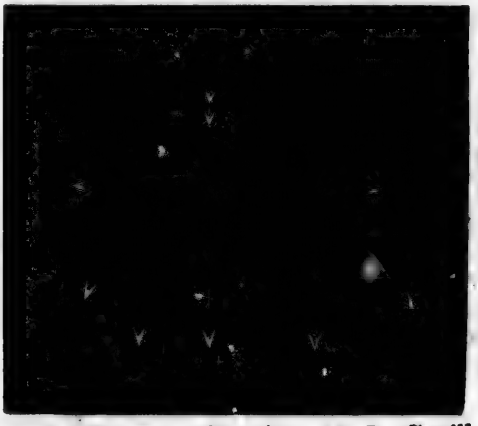
Where, as in the figure, the two forces tend to turn the system in opposite directions, their moments will bear different signs, and we have, by the same process as above,

$$R.OD = P.OB - Q.OC;$$

and the direction in which the system tends to turn will be indicated by the sign of the moment $R.\ OD$ found from this expression.

35. *Two Forces act in parallel directions on a rigid system.* Two parallel Forces are equivalent to a single Force

Let P, Q be the two Forces; O, any point in their plane.



Draw $O\,C\,B$ perpendicular to the forces. At B, C, apply two equal and opposite forces T; these will in no way affect the system.

Let R be the Resultant of P, T, acting at B; and S that of Q, T, at C. Then the directions of R, S will in general meet: let them do so in A, and suppose them to act at this point. They can now be here resolved into their original components, parallel to them.

whose magnitude is their sum

P, T; Q, T; of which the forces T, being equal and opposite, may be removed altogether, leaving the forces P, Q acting in a direction parallel to their original direction, and combining into a single force $(P + Q)$.

Again, the moment of this single force $(P + Q)$ about O is equal to the algebraic sum of those of its components R and S (§ 34); and the moment of R is equal to the sum of the moments of its components, namely, P and T; and so is that of S to the sum of the moments of Q and T; among which moments those of the forces T destroy each other, leaving the algebraic sum of the moments of the original forces P, Q equal to the moment of the single force $(P + Q)$, which has been shown to be their equivalent.

and whose moment is the sum of their moments.

If the Forces P, Q had been taken acting in opposite directions, we should have found by the same process that the single equivalent force had for its magnitude the difference of those of P and Q, and acted in the direction of the greater force, but that its moment was still equal to the algebraic sum of their moments.

If, therefore, we now extend to parallel forces the same method of indicating oppositeness of direction by difference of sign, which was used in the case of Forces acting in the same line, we can include the above cases in a single statement, as follows:

Two parallel forces acting on a rigid system are equivalent to a single parallel force which is equal to their algebraic sum, and whose moment round any point in the plane of the forces is equal to the algebraic sum of the moments of the two forces.

Exception. A couple.

In one case, however, the above process becomes nugatory, which is when the two forces are parallel, equal, and opposite. Such a pair of forces is called a *couple*, and the case must be excluded from our general statement.

Any two Forces, Resultant of.

36. If to this single equivalent Force in § 35, we give the name of *Resultant*, we can now include the results of the two last articles in one statement:

Any two Forces in the same plane acting on a rigid system (unless they form a couple) are equivalent to a single Resultant force, whose moment round any point in their plane is equal to the algebraic sum of their moments round this point.

Any Forces in one plane

37. *Any Forces act in one plane on a rigid system.*

Taking any two of these, we find their Resultant, its moment being the sum of the moments of the two round any assumed point in the plane; combining this Resultant with a third to form a new Resultant, whose moment will be the sum of those of the three forces; and this again with a fourth; and so on till we have taken all the forces, we are left at last with a single Resultant only, whose moment is equal to the sum of the moments of all the Forces. In thus proceeding, we must avoid combining with any one of the partial resultants a force which would form with it a couple; and this we can always do by taking instead of this force another one which will *not* form a couple, for if it did, there would then be *two* equal and parallel forces, *not* opposite, and these two could be combined into one which would now no longer form a couple with the Resultant spoken of; we can thus always evade forming a couple until we have combined all the forces but one, and it may happen that this one is equal, parallel, and opposite to the Resultant we have obtained from all the rest, so that we have a *couple* remaining.

are reducible to single Resultant, in general,

Hence, *any set of Forces acting in one plane on a rigid system are either reducible to a couple, or else to a single Resultant Force, whose moment round any point in the plane is equal to the algebraic sum of the moments of the Forces round that point..*

38. *To find the conditions of equilibrium when Forces in one plane act on a rigid system which can turn freely about a fixed point in that plane.*

and for equilibrium

The forces are reducible either to a couple or to a single Resultant Force.

In the former case, equilibrium is not possible; in the latter, equilibrium will subsist if the Resultant either be zero or pass through the fixed point, and each of these suppositions will make its moment about this point vanish, and therefore also the algebraic sum of the moments of all the Forces, to which sum it has been shown to be equal.

Hence the necessary and sufficient condition of equilibrium is, that

e sum of eir moments vanishes.

The algebraic sum of the moments of all the Forces about the fixed point must vanish.

This principle will always be quoted by the name of "the vanishing of moments."

39. The same principle may easily be seen to apply when the rigid body is capable of turning about a fixed straight line or axis, and the forces are not all in one plane but are perpendicular to this axis. The moment of each Force being taken about that point of the axis which is cut by a perpendicular plane containing the Force, we can state the condition of equilibrium in the form:

The algebraic sum of the moments of the Forces about the fixed axis must vanish.

CHAPTER IV.

CENTRE OF PARALLEL FORCES, AND OF GRAVITY.

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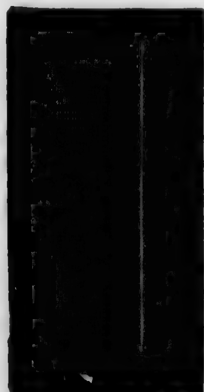
40. It has been shown that two parallel forces (not forming a couple) acting on a rigid system, have for Resultant a single force in the same plane; its direction being parallel to that of the two, its magnitude being the algebraic sum of their magnitudes, and its moment about any point in this plane being the algebraic sum of their moments about this point.

Resultant of two parallel Forces has its

moment about a perpendicular line

Also, the moment of a force about a line to which its direction is perpendicular has been defined to be the product of the number expressing the magnitude of the Force, by the perpendicular distance between its direction and the line.

41. It will now be shown that the moment of this Resultant of two parallel forces, *about any line* perpendicular to their direction, *is* equal to the algebraic sum of the moments of the two forces about this line.



Suppose the forces P, Q, to be acting in the same direction perpendicularly to the plane of the figure and meeting this plane in the points B, C; and their Resultant R (which $= P + Q$, and acts parallel to and in the same plane with them), to meet the plane of the figure in A. Let $b\,a\,c$ be any line in this plane, and draw to it the perpendiculars Aa, Bb, Cc. Then, if $B\,A\,C$ be parallel to $b\,a\,c$, Aa, Bb, Cc, are all equal, and the proposition is manifestly true, for

equal to the sum of the

$$R.\,Aa = (P + Q)\,Aa = P.\,Bb + Q.\,Cc.$$

But if BAC be not parallel to bac, let them meet in O. Then, O being a point in the plane of the forces, the moment of R round O is equal to the sum of those of P and Q round it, and therefore

$$R.\ OA = P.\ OB + Q.\ OC.$$

But by similar triangles

$$\frac{Aa}{OA} = \frac{Bb}{OB} = \frac{Cc}{OC}$$

moments of the Forces.

and therefore

$$R.\ Aa = P.\ Bb + Q.\ Cc,$$

which proves the proposition for this case, and the proof holds good also for the cases where the forces or their moments are in opposite directions, having due regard to algebraic sign.

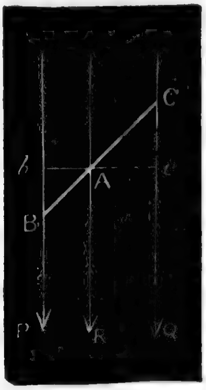
Any number of parallel Forces have in general a single Resultant.

42. *Any parallel Forces, acting on a rigid system, are either reducible to a couple or else to a single Resultant Force which acts in a parallel direction, its magnitude being the algebraic sum of the magnitudes of all the Forces, and its moment about any assumed line perpendicular to their direction, being equal to the algebraic sum of the moments of all the Forces about this line.*

Taking any two of the forces (which do not form a couple), we find their Resultant, which acts in a parallel direction, its magnitude being the algebraic sum of their magnitudes, and its moment, about any assumed line perpendicular to the direction of the forces, being equal to the algebraic sum of their moments about this line: combining this Resultant with any third force to form a new Resultant, and this again with a fourth, and so on as in § 37, we arrive at last either at a couple or a single Resultant Force acting in a parallel direction, its magnitude being equal to the algebraic sum of the magnitudes of all the Forces, and its moment about the assumed line being equal to the algebraic sum of all their moments about this line.

43. *The centre of Parallel Forces.* Centre of parallel Forces.

When given parallel Forces, acting at given points of a rigidly connected system, are reducible to a single Resultant, its direction passes through a point whose position is invariable with regard to the points of the system, whatever be the direction of the Forces.



Take any two of the Forces P, Q, acting at the given points B, C. Join BC and let their resultant R cut it in A. Then, the moment of R about any point in the plane being equal to the algebraic sum of the moments of P, Q, about this point; let these moments be taken about A.

The moment of R about A is zero; hence, drawing $b\,A\,c$ perpendicular to the direction of the forces,

$$P.\,Ab - Q.\,Ac = 0.$$

and, by similar triangles,

$$\frac{AB}{AC} = \frac{Ab}{Ac}, \text{ and therefore,}$$
$$= \frac{Q}{P}.$$

Hence $B\,C$, which is given, is cut in the point A in a ratio which is independent of the direction of the forces with regard to $B\,C$, and the position of A is therefore *given* with regard to B and C.

Now, taking any third force, acting at D, we may combine it with the resultant of P and Q, and the point in which the new resultant cuts $A\,D$ will be given in position with regard to A and D or to A, B and C.

And thus we may go on till we arrive at the final resultant.

Hence, the proposition as enunciated is true.

This point is called the *centre of Parallel Forces.*

The system may be turned about it.

44. If this point—the centre of parallel forces—in a given system be rigidly connected with the system and supported or fixed, the system will be kept at rest, and will remain so when the forces are turned about their points of action into any other direction. It will also still be at rest if it be turned about this point into any other position, the forces acting always at the same points of the system and being always parallel to each other, though their directions may be varied at pleasure.

The pressure supported by this fixed centre is evidently the algebraic sum of the forces, and the algebraic sum of their moments about any line through this point vanishes.

45. *The Centre of Gravity.*

Centre of Gravity.

When the only forces acting on a system are the weights of the several particles of that system, if we suppose the vertically-downward directions in which these weights act to be parallel to each other, and the weight of any particle to be independent of its position; then, since the forces all act in the same direction, they have a single Resultant which is equal to their sum, that is, to the weight of the whole system, and acts vertically downwards through the *centre of parallel forces*, which is in this case called the *centre of gravity*.

Whole weight may be collected at:

46. The statical effect, therefore, of any rigid system will not be altered by supposing it to be without weight, and the whole weight to be collected at its centre of gravity and there to act—this point, however, being rigidly connected with the system.

We may also, without alteration of the statical effect, conceive the system to be geometrically divided into any number of systems, and the weight of each of these to be collected at its own centre of gravity and there act, these partial centres of gravity being rigidly connected with each other and the system.

System balances about in all positions:

47. Also, if the centre of gravity of a system be supported or fixed, the system will balance about this point in all positions under the sole action of the weights of the parts of the system, these being rigidly connected with each other and the centre of gravity, and this is sometimes made the definition of the *centre of gravity*.

How found.

48. The position of the centre of gravity relative to a given system will be determined from the consideration, that, placing the system so that any given line in it shall be horizontal, and equating the moment of the whole weight collected at the centre of gravity with the moments of the several weights of the particles about this line, the distance of the centre of gravity from the vertical plane passing through this line will be found. Taking thus three planes in succession intersecting in a point, the distances of the centre of gravity from each of these planes can be found, and its position therefore determined.

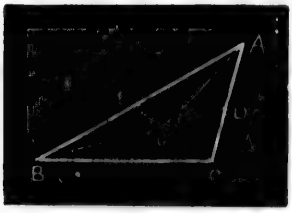
49. Since the position of the centre of gravity in the system depends on the relative and not the absolute weights of its parts, this position will not be affected by increasing or diminishing proportionally these weights.

Of a uniform body.

50. If a rigid body be of uniform density: that is, if the weight of a given volume of its substance be the same in every part of the body; then, if there be a line about which the form of the body is symmetrical, the centre of gravity will be in that line; and if there be two such lines, the centre of gravity will be their intersection. Thus the centre of gravity of a circle or sphere is the centre; of a parallelogram or parallelopiped, the intersection of its diagonals; of a regular prism or cylinder, the middle point of its axis.

51. If a uniform body balance in every position about a line, the centre of gravity lies in that line; and if about two such lines separately, it will be their intersection. Thus a triangular area will balance about a line drawn from one angle to bisect the opposite side, for the triangle can be generated by a line moving parallel to one side, and the small area generated at any stage of its motion will balance about the line

Of a triangular area. which bisects it. Hence the centre of gravity of a triangular area is the intersection of lines drawn from the angles to bisect the opposite sides, and this intersection is at a distance from the angle of two-thirds of the bisecter drawn from it.



For let $A\,B\,C$ be the triangle, and $B\,D$, $C\,E$ bisect $A\,C$, $A\,B$, and meet in G. Then G is the centre of gravity.

Join $E\,D$, which is parallel to $B\,C$ (Eucl. B. VI. 2),

$$\text{Then } \frac{B\,G}{G\,D} = \frac{B\,C}{E\,D}, \text{ by similar triangles } B\,G\,C,\ E\,G\,D$$

$$= \frac{C\,A}{A\,D}, \text{ by similar triangles } A\,C\,B,\ A\,D\,E$$

$$= \frac{2}{1}.$$

Hence $B\,G$, being double of $G\,D$, is $\frac{2}{3}\,B\,D$.

The same point G is also the centre of gravity of three equal bodies placed at the points A, B, C.

Of any polygonal area. Cor. In this way can the centre of gravity of any polygonal area be found; for, dividing the figure into triangles, the weight of each of these may be supposed collected at its own centre of gravity, and the centre of gravity of the whole figure will be that of these weights, considered as heavy particles situated in those points.

The method of finding this latter will be treated in the following article.

Of any heavy particles in one plane. 52. *To find the centre of gravity of a system of particles all in one plane.*

Let Ox, Oy be two perpendicular lines in this plane, with regard to which the positions of the particles are known.

Let P be the place of one of the particles, w its weight.

Draw $P N$, $P M$ perpendicular to Ox, Oy, respectively, and denote $P M$ by x, $F N$ by y.

Suppose the plane of the figure to be horizontal; then Ox is a line perpendicular to the direction of the weights, and therefore (§ 36) the moment about Ox of the whole weight collected at the centre of gravity is equal to the algebraic sum of the separate moments about it. Hence if W be the whole weight, and the distance of the centre of gravity from Ox be denoted by $\bar{y}$, we have

$$W.\bar{y} = \Sigma\,(w.\,y)$$

and

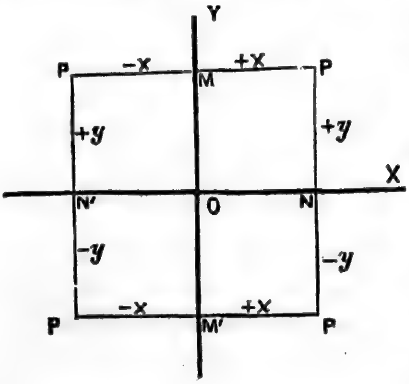
$$\bar{y} = \frac{\Sigma\,(w.\,y)}{W}$$

where Σ denotes the algebraic sum of all the products corresponding to that within the bracket. Also, if a moment be reckoned positive when P is above the line Ox, it will plainly be negative when P is below the line, and the difference in sign of the moments will therefore at once be indicated by considering a y positive when drawn upwards from Ox, and negative when downwards.

Similarly, by taking moments round Oy, if $\bar{x}$ be the distance of the centre of gravity from Oy, we have

$$\bar{x} = \frac{\Sigma\,(w.\,x)}{W}$$

where x will be considered positive when drawn to the right of Oy, negative when to the left.



The distances of the centre of gravity from these two lines being thus found, and the directions in which these distances are drawn being indicated by the signs with which they are affected, the position of the centre of gravity is fully determined.

Of particles in a straight line.

Cor.—*If the particles all lie in the same line, take this for* Ox. Then, every y being zero, $\overline{y}$ is so also, and the centre of gravity is in Ox, its distance from O being given by

$$\overline{x} = \frac{\Sigma (w\, x)}{W}$$

The following independent proof of this may be noted.

53. Let Ox be the line in which the particles lie, O being any point from which the distances of the particles are known, and let this line be placed horizontal. Let x be the distance from O of the particle whose weight is w.

Let W be the whole weight, and $\overline{x}$ the distance of the centre of gravity from O.

Draw another horizontal line from O perpendicular to Ox. This line will then be perpendicular to the direction of the weights, and the moment about it of the whole weight collected at the centre of gravity will be equal to the algebraic sum of the moments of the several weights. Hence we have

$$W.\overline{x} = \Sigma\,(w.x)$$

or

$$\overline{x} = \frac{\Sigma\,(w.x)}{W}$$

where Σ denotes the algebraic sum of all the products corresponding to that within the bracket. Also, if a moment be reckoned positive when the particle is on one side of O, that of a particle on the other side of O will be negative, and the difference in algebraic sign of the moments will therefore at once be indicated by considering the x's of the particles to be positive or negative according as they lie on one or the other side of O.

Heavy body suspended freely, its centre of gravity is vertically above or below the point of suspension.

54. *When a rigid body rests suspended from or supported by a fixed point, and acted on only by its weight, the vertical line drawn through the centre of gravity will pass through the point of suspension or support ; and, conversely.*

For the weight of the body may be supposed collected at its centre of gravity, and there to act vertically downwards; and the necessary and sufficient condition of equilibrium is that its moment about the fixed point must vanish, which requires that its direction shall pass through this point.

Cor. 1.—When the centre of gravity is vertically *under* the point of support or suspension, if the body be slightly disturbed from rest, the moment of the weight will tend to bring it back again to its original position. The equilibrium is therefore said to be *stable*. When the centre of gravity is vertically *above*, the contrary takes place, and the equilibrium is *unstable*.

Cor. 2.—This affords a practical method of finding the centre of gravity of any plane area. Thus, suspending it freely from any one point, trace on it, when at rest, the direction of the vertical passing through this point: then, taking any other point (not in this line) for a new point of suspension, trace also the vertical through it. The intersection of the two lines thus drawn is the centre of the gravity required.

Body placed on a horizontal plane, will it stand or fall over

55. *When a rigid body, having a plane base, is placed with this in contact with a fixed horizontal plane, and is acted on only by its own weight, it will stand or fall according as the vertical through its centre of gravity passes within or without the base.*

By the base is here meant the figure included by a string stretched completely round the outside of the plane section of the body which is in contact with the horizontal plane.

If the body fall over, it must begin to turn round some tangent to the curve formed by this string, and the moment of the weight, supposed collected at the centre of gravity, must tend about this tangent in a direction *from* the inside *towards* the outside of the area of the base, and the vertical through the centre of gravity will pass outside the base.

Also, when this vertical passes outside, the body must fall over; but if this vertical pass within the base, the moment of the weight about *every* tangent to the string tends in direction from the outside towards the inside, and the body cannot fall over.

CHAPTER V

THE MECHANICAL POWERS.

he Machines.

56. It is usual to treat of the *Simple Machines*, or *Mechanical Powers* as they are sometimes called, under six classes, namely—the Lever, the Wheel and Axle, the Pullies, the Inclined Plane, the Screw, and the Wedge. Of these, the Wedge will not be here considered, as in its practical application the investigation on the principles of the foregoing chapters would be of small utility.

Mechanical advantage defined.

When a power P sustains on any one of these machines a weight W, the ratio $W : P$ is called the *mechanical advantage* of the machine; and the machine is said to gain or lose advantage according as this ratio is greater or less than unity.

In the following investigations, bodies will be supposed rigid, surfaces smooth, strings perfectly flexible and of insensible size, and the parts of the machine to be without weight, unless otherwise specified.

57. *The Lever.*

Straight Lever.

Archimedes and Da Vinci.

A straight lever is a rod capable of turning freely in one plane about a point in itself which is fixed. This fixed point is called the *fulcrum*.

Fig. 1.

CASE 1.—The weight W at one end of the lever supported by a weight P at the other end.

$B\ A\ C$ the lever; A the fulcrum.

Draw $b\ A\ c$ horizontal, and therefore perpendicular to the direction of the weights.

Then by the vanishing of moments,

$$P.\ Ab - W.\ Ac = 0$$

or

$$P.\ Ab = W.\ Ac$$

But, by similar triangles, $\dfrac{A\,B}{A\,b} = \dfrac{A\,C}{A\,c}$

and therefore $\quad P.\ A\,B = W.\ A\,C.$

Cor. 1.—The pressure on the fulcrum is the weight $(P + W)$ acting vertically downwards.

Cor. 2.—Since the relation $P.\ AB = W.\ AC$, does not involve the angle at which the lever is inclined to the horizon, it follows that if the lever be at rest in any one position (except a vertical one), on being turned into any other position it will still be at rest.

Case II.—The power P and the weight W acting in opposite (but parallel) directions, and the weight nearer to the fulcrum than the power. Fig. 2.

Using the same construction and reasoning as in the former case, we have here also

$$P.\ AB = W.\ AC.$$

Cor. 1.—The pressure on the fulcrum is here $W - P$, acting vertically downwards. The second corollary also holds.

Case III.—The power P and the weight W acting in opposite but parallel directions, and the power nearer to the fulcrum than the weight. As before, we have Fig. 3.

$$P.\ AB = W.\ AC.$$

Cor. 1.—The pressure on the fulcrum is $P - W$ and acts vertically upwards. The second corollary also holds.

58. In all these cases, the mechanical advantage $\left(\dfrac{W}{P}\right)$ is $\dfrac{A\,B}{A\,C}$ Mech. adv.
or the ratio of the arms of the power and weight. In Case I.

this ratio may be either equal to, greater, or less than unity; but in Case II. it is always greater, and in Case III. less: hence, advantage is always gained in the second case and lost in the third, but may be either gained or lost in the first.

Lever supposed heavy.

59. If the weight of the lever (w) be taken into account, it may be supposed collected at the centre of gravity G (which will be the middle point if the lever be uniform).

Let the vertical through G meet the horizontal $A\,b\,c$ in g.

Then, by the vanishing of moments about A,

$$P.\,Ab + w.\,Ag - W.\,Ac = 0$$

but by similar triangles,

$$\frac{A\,B}{A\,b} = \frac{A\,G}{A\,g} = \frac{A\,C}{A\,c}, \text{ and therefore}$$

$$P.\,A\,B + w.\,AG - W.\,AC = 0$$

or,

$$P.\,A\,B + w.\,AG = W.\,AC$$

Similarly, in Cases II. and III. we should find

$$P.\,A\,B = W.\,A\,C + w.\,AG.$$

Here also the lever will balance in all positions about the fulcrum.

Common Balance.

60. In the common Balance, which consists of a heavy beam, having scale pans suspended at its ends, and balancing about a horizontal knife edge, the pans and arms of the beams are made perfectly equal and similar on each side of the edge, but the centre of gravity of the beam is made to fall vertically below the knife-edge when the beam is horizontal. The beam will therefore rest in a horizontal position only when the pans are loaded with equal weights; and if then disturbed from this position, the moment of its own weight brings it back, so that the equilibrium is *stable*.

Roman Steelyard. Fig. 4.

61. In the common or Roman Steelyard, a heavy beam has attached to it a knife-edge which is supported as a fulcrum; a weight runs along the upper straight edge of the beam on the longer arm, and the substance to be weighed is attached at a fixed point to the shorter arm by a hook or scale-pan. The longer arm is graduated, and the weight of the substance is known from the graduation at the point where the moveable weight is, when the beam is at rest.

In fig. 4, A is the knife-edge or fulcrum, P the weight moveable along AB: C, the point whence the substance, whose weight W is required, is suspended.

CAB being horizontal, let O be the point on the other side of A where P would keep the Steelyard at rest when the weight W is away. The moment of the weight of the Steelyard about A is therefore equal to $P.\,AO$.

Now let the weight W be attached, and let M be the place of P when equilibrium is obtained. Then, taking moments about A,

$$\begin{aligned} W.\,AC &= P.\,AM + \text{moment of weight of Steelyard} \\ &= P.\,AM + P.\,AO \\ &= P.\,OM \end{aligned}$$

$$\text{Hence, } W = \frac{P}{AC}\,OM,$$

And, since P and AC are invariable, W is proportional to OM: O therefore is the point from which the graduation must be made. Thus, if P be at B_1 when W is 1 lb, and we take $OB_1 = B_1B_2 = B_2B_3 = \ldots$; then when P is at B_2, B_3, ... W will be 2, 3, ... lbs.

62. The preceding cases of the lever are only special applications of the general investigation in Chap. III. In fact, any body moveable about a fixed point and acted on by forces in the plane of that point may be considered a *lever*, and the principle of § 38 is often quoted as the *principle of the lever*. "Principle of the lever."

63. *The Wheel and Axle.*

Wheel and axle Fig. 5.

This machine consists of a circular drum or wheel, which is attached to a cylinder or axle, its centre lying in the axis of the cylinder and its plane being perpendicular to this axis. The whole system runs freely on this axis, which is fixed; and the power P acts by a string coiled round the wheel, and supports a weight W which hangs from a string coiled round the axle. The strings being perpendicular to the axis, and also to the radii of the wheel and axle respectively at the points where they become uncoiled, we have for the condition of equilibrium, by § 39, taking moments about the axis,

$$P \times \text{radius of wheel} - W \times \text{radius of axle} = 0,$$

Mech. adv. Hence the mechanical advantage $\left(\frac{W}{P}\right)$ is equal to the ratio of the radii of the wheel and axle.

Cor.—Any number of wheels and axles may run on the same axis, and the condition of equilibrium will be that the sum of the products of each power into the radius of its wheel is equal to the corresponding sum for the weights and radii of the axles, the powers being all supposed to turn in the *same* direction and the weights all in the opposite.

Pullies 64. *The Pullies.*

A pully is a wheel running freely on an axis, which, passing through its centre, is fixed in a block by which the pully is suspended and to which a weight may be attached. The circumference of the pully is grooved to admit of a string passing over or under it. The pully is said to be *fixed* or *moveable* according as its block is so.

Single fixed Pully.

Let P, Q be the forces, applied at the ends of the string passing over the pully. The whole system being smooth, the tension of the string is the same throughout (§ 10), and, therefore,

$$P = Q.$$

No mechanical advantage is gained or lost.

Single moveable Pully. Fig. 6. 65. *Single moveable pully supported by a string passing under it, the free portions of the string being parallel, and a weight attached to the block.*

Let P be the force applied to the string on one side of the pully; then, the whole system being smooth, the tension of the string is the same throughout, and P is therefore also the force applied to the string on the other side. There are then two parallel forces, each equal to P, supporting a weight W which acts vertically. Hence the strings must be vertical, and

$$W = 2\ P.$$

Here the mechanical advantage is 2.

Cor. 1. If one end of the string be attached to a fixed point, the same result holds good, for the tension must be the same throughout.

Cor. 2. If the weight w of the pully, including the block, be taken into account, it may be supposed attached to the block, and we have

$$W + w = 2 P.$$

65. *First system of pullies.* First system Fig. 7.

A number of pullies are attached to the same block, which supports a weight, and the same string passes round all the pullies.*

The portions of the strings between the pullies are supposed to be parallel, and will therefore also be vertical as in § 65. Let P be the force applied to the string; P will then be the tension throughout, and the weight W is supported by as many parallel forces, each equal to P, as there are parallel portions of the string at the lower block; and the number of these portions is evidently double the number of pullies at this block. Hence, if n be the number of moveable pullies,

$$W = 2 n P;$$

and the mechanical advantage is $2n$.

Cor. If the weight of all the pullies within the lower block be w, the weight of the block also being included, we may suppose this weight attached to the block, and

$$W + w = 2 n P.$$

67. *Second system of pullies.* Second system. Fig. 8.

Each pully hangs by a separate string, the last pully supporting the weight; the free portions of all the strings are parallel, and therefore vertical.

Let A_1, A_2, A_3,... be the pullies, n being the number of them; W, the weight supported at the last one; P, the power applied at the first string. Number the strings 1, 2, 3,... according to the pully under which each passes. The tension of each separate string is the same throughout.

The tension of (1) is P; the weight supported at A_1 is double the tension of (1), and therefore $= 2\ P$, and this is the tension of (2).

The weight supported at A_2 is double the tension of (2) and therefore $= 2\ (2\ P) = 2^2\ P$, and this is the tension of (3).

The weight supported at A_3 is double the tension of (3), and therefore $= 2\ (2^2\ P) = 2^3\ P$, and this is the tension of (4).

Proceeding in this way, we come at last to the weight supported at $A_n = 2^n\ P$, and this is the attached weight. Hence,

$$W = 2^n\ P,$$

and the mechanical advantage is 2^n.

Cor. 1. The mechanical advantage is doubled by every additional pully.

Pullies supposed heavy

Cor. 2. The weight of the pullies may be readily taken into account by observing that, from the preceding, the force required to support a weight W on n moveable pullies is $\frac{W}{2^n}$.

Let w_1, w_2, w_3, ... be the weights of the several pullies, blocks included. Each of these weights may be supposed a weight attached to its block, and supported on the system of pullies above it.

The power required to support			w on one	moveable pully	is	$\frac{w_1}{2}$.
"	"	"	w_2 on two	"	pullies is	$\frac{w_2}{2^2}$.
"	"	"	w_3 on three	"	"	$\frac{w_3}{2^3}$.
"	"	"	w_n on n	"	"	$\frac{w_n}{2^n}$.
Also	"	"	W on n	"	"	$\frac{W}{2^n}$.

And the whole power required will be the sum of these; therefore,

$$P = \frac{w_1}{2} + \frac{w_2}{2^2} + \ldots + \frac{w_n}{2^n} + \frac{W}{2^n}, \text{ or}$$

$$W = 2^n P - \left(2^{n-1} w_1 + 2^{n-2} w_2 + \ldots + w_n\right)$$

The weight of the pullies therefore lessens the advantage of the machine.

Cor. If the weight of each pully be the same (w), then

$$W = 2^n P - (2^{n-1} + 2^{n-2} + \ldots + 1)\, w$$

$$= 2^n P - (2^n - 1)\, w.$$

68. *Third system of pullies.* Third system, Fig. 9.

Each pully hangs by a separate string which is attached to a bar or block carrying the weight, and the free portions of all the strings are parallel, and therefore vertical.

This is the second system turned upside down, the weight becoming a fixture, and the beam to which the strings are attached becoming a moveable bar carrying a weight, and the mechanical advantage might be inferred from the preceding. The pressure supported by the beam in the second system is the sum of the tensions of the strings, that is,

$P + 2P + 2^2 P + \ldots$ to n terms, $= (2^n - 1) P$, and this becomes the weight W in the third system. Therefore,

$$W = (2^n - 1) P.$$

The last pully (A_n), however, becomes fixed, so that the number of moveable pullies is only $(n - 1)$. Making n the number of *moveable* pullies we have

$$W = (2^{n+1} - 1) P.$$

The following is an independent investigation for this case.

Let $A_1, A_2, A_3, \ldots A_n$, be the pullies, n being their number exclusive of the last one A, which is fixed, and $\overline{n + 1}$ the number of strings.

B_1, B_2, B_3, ... B_{n+1}, the points at which the respective strings are attached to the straight bar which carries the weight W. Number the strings 1, 2, 3, ... according to the pully over which each passes.

The tenison of each separate string is the same throughout.

The weight supported is the sum of the pressures of the strings at B_1, B_2, B_3. ...

The tension of (1) is P, and this is the pressure at B_1.

The weight supported at A_2 is double the tension of (1) and $= 2P$, and this is therefore the tension of (2) and the pressure at B_2.

The weight supported at A_3 is double the tension of (2) and $= 2(2P) = 2^2 P$; and this is therefore the tension of (3) and the pressure at B_3.

Proceeding in this way we obtain the tension of the $(n+1)$th string and pressure at $B_{n+1} = 2^n P$.

Taking the sum of all these pressures,

$$W = P + 2P + 2^2 P + \ldots\ldots + 2^n P$$
$$= (2^{n+1} - 1) P,$$

and the mechanical advantage is $2^{n+1} - 1$.

Pullies supposed heavy.

Cor. The weights of the pullies may be taken into account by observing that each may be considered as a power acting by means of the string from which it hangs, and supporting a weight on the system of moveable pullies above it.

Let w_1, w_2, w_3, ... w_n, be the weights of the pullies, blocks included.

The weight supported by w_1 on $(n-1)$ moveable pullies is $(2^n - 1) w_1$.

"	"	w_2 on $(n-2)$	"	" $(2^{n-1} - 1) w_2$.
"	"	w_n on 0	"	" $(2 - 1) w_n$.
Also "	"	P on n	"	" $(2^{n+1} - 1) P$.

The whole weight W (including that of the bar) is the sum of these; therefore,

$$W = P(2^{n+1}-1) + w_1(2^n - 1) + w_2(2^{n-1}-1) + \ldots + w_n(2-1).$$

The weight of the pullies therefore increases the advantage of the machine.

Cor. 1. If the weight of each pully be the same (w), then,

$$W = P(2^{n+1}-1) + w(2^n + 2^{n-1} + \ldots + 2 - n)$$

$$= P(2^{n+1}-1) + w(2^{n+1} - 2 - n).$$

If we put $P = 0$, we have

$$W = w(2^{n+1} - 2 - n),$$

which is the weight that would be supported by the pullies alone.

Cor. 2. The point of the bar to which the weight should be attached in order that the bar may be horizontal will be the *centre of parallel forces* for the tensions of the strings and the weight of the bar. If we neglect the weight of the pullies and the bar, this point will remain the same in a system, whatever be the power; if, however, the weight of bar and pullies be considered, it will be different for different powers.

69. Taking the same number n of moveable pullies in each system, the respective mechanical advantages are $2n$, 2^n, $2^{n+1}-1$, and these numbers are in ascending order of magnitude. Hence the mechanical advantage of the third system is greater than that of the second, and of the second than of the first when there is more than one pully. Systems compared.

70. The following combination of pullies may be noticed. It is called the *Spanish Barton*. Spanish Barton. Fig. 10.

The tension of the string to which P is attached is the same throughout and $= P$. That of the other string is also the same throughout and $= 2P$. Therefore $W = 4P$.

If we take the weights of the pullies A, B into account, we have $W + B = 4P + A$.

Inclined plane.

71. *The Inclined Plane.*

This is a plane fixed at a certain angle (called its *inclination*) to the horizon, and on it a heavy particle is supported by a force applied and the reaction of the plane. Since the plane is smooth, its reaction is exerted in a normal direction; also the weight of the particle acts vertically: therefore if a vertical plane be drawn through the particle and the normal to the inclined plane, since the plane thus drawn contains the directions of those two forces acting on the particle, the third force or *Power* must also act in this plane.

Fig. 11. Let th figure represent this plane; AB, the section of the inclined ane; AC, horizontal.

The angle BAC is the inclination, α (suppose).

Let P, the power, act at an angle θ to AB, and let

R be the reaction of the plane exerted perpendicularly to AB.

W the weight of the particle acting vertically downwards.

The particle is then kept at rest by the three forces P, R, W.

Taking the resolved parts of these along AB, that of P is $P\cos\theta$; of R is 0; of W is $W\cos(90°-\alpha) = W\sin\alpha$.

Hence by the "vanishing of the Resultant,"

$$P\cos\theta - W\sin\alpha = 0,$$

which gives the mechanical advantage $\left(\frac{W}{P}\right) = \left(\frac{\cos\theta}{\sin\alpha}\right)$.

Cor. 1. For a given inclination, the mechanical advantage is greatest when $\cos\theta$ is greatest; that is, when $\theta = 0$, and the force acts parallel to the plane.

For a force acting at a given angle to planes of different inclinations, the mechanical advantage increases as the inclination diminishes.

Cor. 2. Resolving the forces horizontally we have

$$P \cos (\theta + \alpha) - R \sin \alpha = 0.$$

Also resolving them perpendicularly to the direction of P,

$$W \cos (\theta + \alpha) - R \cos \theta = 0,$$

These two equations give R in terms of P or W.

Or these relations might at once have been asserted from the "triangle of forces" (§ 29): for this gives

$$\frac{P}{\sin \alpha} = \frac{W}{\cos \alpha} = \frac{R}{\cos (\theta + \alpha)}$$

Although these results have been obtained only for a particle, they are true for a body of finite size supported on the plane by a power whose direction passes through its centre of gravity.

72. In the case where the power is acting parallel to the plane, as where it is exerted by a string, parallel to the plane, passing over a pully and supporting a weight P hanging freely, we have from the above by putting $\theta = 0$, or at once by resolving the forces along AB, observing that P is the tension of the string, Power acting parallel to plane. Stevinus.

$$P - W \sin \alpha = 0$$

and the mechanical advantage $\dfrac{W}{P} = \dfrac{1}{\sin \alpha}$, and is the cosecant of the inclination.

If BC (vertical) be called the *height* of the plane, AB its length: then, since $\sin \alpha = \dfrac{BC}{AB}$, we have

$$P - W \frac{BC}{AB} = 0$$

or $$\frac{P}{W} = \frac{\text{height}}{\text{length}}$$

Again, resolving forces perpendicularly to the plane, we have

$$R - W \cos a = 0$$

or $$R = W \cos a = W \frac{AC}{AB}$$

and $$\frac{R}{W} = \frac{\text{base}}{\text{length}}$$

Hence the power, weight and pressure on the plane are proportional to the height, length and base of the plane.

Cor. This latter result is at once seen from the "triangle of forces;" for, drawing CN perpendicular to AB, the sides of the triangle BCN, taken in order, are parallel to the directions of the forces, and therefore represent them in magnitude; and the triangle ABC is similar to BCN.

Screw. 73. *The Screw.*

Fig. 14. The Screw is a circular cylinder, on the surface of which runs a protuberant spiral thread, whose inclination to the axis of the cylinder is everywhere the same. This thread works freely in a fixed block, wherein has been cut a corresponding groove. The power is applied perpendicularly to a rigid arm which passes perpendicularly through the axis of the cylinder and is rigidly attached to it, and the weight is supported on the cylinder (whose axis is here supposed to be vertical), and may be supposed to act in the direction of this axis.

74. The complement of the invariable inclination of the thread to the axis, or (the axis being vertical) the inclination to the horizontal line which touches the cylinder at the point, is called the *pitch* of the screw. If a right angled triangle BAC be drawn, having the base AC equal to the circumference of the cylinder, and the angle BAC equal to the pitch of the screw (a), and this triangle be wrapped smoothly on the cylinder, its hypothenuse will mark on the cylinder the course of the thread, and by superposing similar triangles the whole

course of the thread may be continued. BC is then the distance between two contiguous threads, and we have

$$\tan BAC = \frac{BC}{AC}$$

$$\text{or } \tan a = \frac{\text{distance between two contiguous threads}}{\text{circumference of cylinder}}$$

75. The Screw is kept at rest by the weight (W) which acts vertically, by the power P which acts horizontally, and by the reactions of the groove on the thread at the various points in contact. Fig. 14

Since the thread is smooth, the reaction at each point of it is normal to the thread; and the angle between the directions of this normal and the axis, being the same as that between the thread and the horizontal tangent which are respectively perpendicular to them, is a, the *pitch*.

If then we resolve this reaction at any point, R (suppose) into two forces; one, vertical, and the other, horizontal and touching the cylinder, the former will be $R \cos a$, and the latter $R \sin a$. Fig. 15

All these vertical portions being parallel, will form a single vertical resultant whose magnitude is $\cos a\, \Sigma(R)$, and this must counterbalance the weight W, since all the other forces are horizontal.

$$\text{Hence} \quad \cos a\, \Sigma(R) = W. \quad \ldots\ldots \quad (1)$$

Again the horizontal portions of the R's tend to turn the cylinder about its axis, and since each acts in a horizontal direction touching the cylinder, the radius of the cylinder is itself the perpendicular distance between the axis and the direction in each case. Hence the moment of one of these ($R \sin a$) is

$$R \sin a \times \text{radius of cylinder,}$$

and the sum of them all is

$$\sin \alpha \times \text{radius of cylinder} \times \Sigma(R),$$

and this must be equal and opposite to the moment of the power, namely, $P \times$ arm of P.

Hence $\sin \alpha \times$ radius of cylinder $\Sigma(R) = P \times$ arm of P. Dividing the sides of this equality by those of the equality (1)

$$\frac{\sin \alpha}{\cos \alpha} \times \text{radius of cylinder} = \frac{P \times \text{arm of } P}{W}$$

$$\text{Hence, since } \frac{\sin \alpha}{\cos \alpha} = \tan \alpha = \frac{\text{distance between threads}}{\text{circumference of cylinder}}$$

$$\text{dist.bet. threads} \times \left(\frac{\text{radius}}{\text{circum.}} \text{ of cylinder}\right) = \frac{P \times \text{arm of } P}{W}$$

$$\text{but} \quad \frac{\text{radius}}{\text{circum.}} = \frac{1}{2\pi}$$

and $2\pi \times$ arm of P is the circumference of the circle which the end of P's arm would describe, and may be briefly called the circumference of P; hence

$$\frac{P}{W} = \frac{\text{distance between contiguous threads}}{\text{circumference of the power}}$$

Mech. adv. and this ratio inverted is the mechanical advantage.

Cor. 1. The mechanical advantage is increased by diminishing the distance between the threads or increasing the arm of the power.

Cor. 2. If the cylinder be heavy, its weight must be included in W. Instead of supporting a weight, the screw may be producing a pressure at its lower end, and in this case the pressure produced will be increased by the weight of the screw. It may also be producing a pressure at its upper end, and then the pressure produced must be diminished by this weight.

Sometimes the screw is fixed and the block moveable, as in the case of a common *nut;* or the groove may be cut in the screw itself, and the thread project in the block.

To all these cases the preceding investigation applies.

76. *The Wedge.* The Wedge.

This is a solid, whose bounding surfaces are two intersecting planes. Most cutting instruments come under this class. It is used generally to separate the parts of bodies, either by blows or a moving pressure, and in this mode of use its investigation belongs to Dynamics. When used to keep open a rift in a body, it acts generally by means of friction, and not by a weight applied to it; it is therefore useless to proceed with its examination on the principles employed hitherto.

VIRTUAL VELOCITIES.

77. If a machine at rest under a power P and a weight W be put in motion, so however that its geometrical relations are unaltered, the space described by the point of application of the power, estimated always in the direction of the Power, is called the *virtual velocity* of the power: and similarly for the weight. Virtual velocities.

The principle of virtual velocities asserts that the product of P by P's virtual velocity is equal to that of W by W's virtual velocity. Principle of

78. This principle is only a special application of a far more general one, which it is not here necessary to examine. We

shall therefore only establish the principle, as above stated, in a few of the more simple cases, by proving that it leads to the relations of equilibrium already found.

The geometrical relations in a machine being such that the spaces described in different displacements are always proportional, it will be only necessary to prove the principle for a particular displacement, and we may select this as convenient.

Straight lever. 79. *The straight lever under weights at its ends.*

Fig. 16. Let the lever BAC be horizontal, and displace it round A into the position B_1AC_1 the directions of P and W meeting BAC in b, c. Then B_1b is P's *virtual velocity*, and C_1c is W's.

The principle then asserts that

$$P.\ B_1b = W.\ C_1c$$

but by similar triangles, $$\frac{B_1A}{B_1b} = \frac{C_1A}{A_1c};$$

hence $P.\ AB = W.\ AC$, the condition found in § 57.

Wheel and axle. 80. *Wheel and Axle.*

Fig. 5. Suppose the machine to make one complete turn; then, the space descended by P is the circumference of the wheel; and by W, that of the axle. The principle then asserts that

$P \times$ circumference of wheel $= W \times$ circumference of axle,

and the circumferences are as the radii: therefore,

$$P \times \text{radius of wheel} = W \times \text{radius of axle},$$

the condition found in § 63.

Pullies. 81. *Pullies.*

In the single fixed pully, the principle is obviously true.

Single pully. In the single moveable pully (fig. 6), let the pully be raised through one inch, then W is raised through one inch, and

one inch of each portion of the string is set free; therefore, P ascends through two inches, and the principle asserts that

$$P \times 2 = W \times 1$$

which is the condition in § 65.

82. In the first system of pullies, let the lower block be raised one inch: then W is raised one inch, and one inch of each portion of the string at the block is set free; therefore, on the whole, $2n$ inches are set free, and this is the space through which P descends. First system. Fig. 7.

The principle then asserts that

$$P \times 2n = W \times 1,$$

the condition found in § 66.

83. In the second system of pullies, let W be raised 1 inch: then A_n rises through 1 inch, and each pully rises through twice as much as the one below it, and P rises through twice as much as the top-pully; therefore on the whole, P's ascent will be 2^n inches; and the principle asserts that Second system. Fig. 8.

$$P \times 2^n = W \times 1,$$

the condition found in § 67.

84. In the third system of pullies let W and the bar be raised 1 inch: then each pully descends through this 1 inch and twice as much as the pully above it, and P descends through 1 inch and twice as much as A_1. Third system. Fig. 9.

The last pully A_n descends through 1: therefore, the last but one descends through $1 + 2 \times 1$

" two	"	"	$1 + 2\,(1 + 2)$ or $1 + 2 + 2^2$
" three	"	"	$1 + 2\,(1 + 2 + 2^2)$ or $1 + 2 + 2^2 + 2^3$

and so on.

On the whole, P " $1+2+2^2+2^3+\dots+2^n$, or $2^{n+1}-1$.

And the principle asserts that

$$P \times (2^{n+1}-1) = W \times 1$$

the condition found in § 68.

Inclined plane. Fig. 12.

85. In the inclined plane, the power acting parallel to the plane, (fig. 12), let W be at the bottom of the plane and be drawn up to the top. Then W's vertical displacement is the height of the plane, and P's descent is its length. The principle asserts that

$$P \times \text{length} = W \times \text{height},$$

the condition found in § 72.

Screw. Fig 13.

86. In the screw, let one complete turn be made. Then the distance moved through by the end of P's arm, estimated always in the direction of P, is the circumference of P; and the space descended by W is the distance between two threads. The principle then asserts that

$P \times$ circumference of $P = W \times$ distance between two threads, the condition found in § 75.

87. Assuming the truth of this principle of *virtual velocities*, it may be conveniently employed to find the mechanical advantage in many machines—as examples, let us take *Roberval's Balance*, *The Differential Axle*, and *Hunter's Screw*,

Roberval's Balance. Fig 17.

88. In *Roberval's Balance* the sides of a parallelogram are connected by free joints with each other and with a vertical axis passing through the middle points of opposite sides; so that the figure is symmetrical about this axis, and the other opposite sides are always vertical. The weights P, W are carried by arms fixed perpendicularly to these latter sides, which arms are therefore always horizontal. If the machine, when at rest, be displaced, one of the weights ascends as much as the other descends, and they are therefore equal.

This result is independent of the particular points of the arms from which the weights depend, and in this lies the convenience of the machine as a Balance.

89. In the *Differential Axle* (fig. 18), two axles of different sizes run fixed together on the same axis, and the weight is supported on these by a pully, whose string is coiled round these axles in opposite directions. If P be raised by a complete turn of the machine, W descends through a space equal to half the quantity of string set free from the axles; that is, through half the difference of the circumferences of the axles; and, the circumferences being as the radii, we have Differential axle. Fig. 18

$$P \times \text{radius of wheel} = W . \tfrac{1}{2} \text{(difference of radii of axles)}.$$

In the common wheel and axle, the power and wheel being given, the mechanical advantage is increased by diminishing the radius of the axle, but this diminution is practically limited by regard to the strength of the axle. In the above machine, the mechanical advantage may be increased indefinitely, by making the axles more nearly of equal size, without too much weakening them.

If the axles were absolutely equal, the mechanical advantage would be infinite, and it is obvious that any weight would be here supported without a power at all.

90. In *Hunter's Screw* (fig. 19), the weight is supported on a smaller screw, which runs in a companion in the interior of a larger screw, the latter passing through a fixed block and being acted on by a power as usual. Hunter's Screw. Fig. 19.

When the power makes a complete revolution, and raises the large screw through the distance between its threads, the smaller screw at the same time descends *in* the large one through the distance between its own threads, and the weight therefore on the whole rises through the difference between the "distances of the threads" in the two screws. Hence,

$P \times$ circumference of $P = W \times$ difference between distances of contiguous threads in the two screws.

The mechanical advantage can therefore be indefinitely increased by making the distance between the threads more nearly the same in each screw. In the common screw, the advantage is increased by diminishing the distance between the threads, but the diminution is practically limited by regard to the strength of the thread.

If each screw had the same distance of threads, the advantage would be infinite, and it is obvious that any weight would be supported without a power at all, the outer screw rising just as much as the inner screw descends *within* it, so that the weight would be stationary.

P.V.V.

In every machine,

91. When a power P is supporting a weight W on any machine, if the machine be set in motion, it will continue to move uniformly so long as its geometrical relations with the power and weight are unaltered; and if s, S be the spaces gone through by the power and weight in any time (that is, their *virtual velocities*) we have $P \times s = W \times S$. Hence a given force acting through a given space for any time will lift the same weight only through a given space, whatever be the machine through which it acts; and if the weight lifted be increased, in the same proportion will the space through which it is lifted be diminished. Also when a given power lifts a weight through a given space, the greater the weight, the greater in the same proportion is the space through which the power must act, and (the motion being uniform) the longer is the time employed. Hence the principle of virtual velocities is sometimes stated in the form, that "in every machine what is gained in power is lost in time."

what is gained in power

is lost in time.

Work done and efficiency.

92. Hence also this product $P \times s$ or $W \times S$ may be considered the *work done* by the machine, and is sometimes termed its *duty;* while with reference to the power, the names of *mechanical efficiency* and *laboring force* have been given.

In this sense, although *advantage* may be gained by a machine, no *efficiency* is gained or (*theoretically*) lost, but it remains the same as if the power were applied directly without the intervention of the machine.

Practically, *efficiency* is always lost, owing to the various resistances due to the parts of the machine.

93. Among engineers the standard of *efficiency* in the comparison of machines has usually been taken to be a *horse power*, which is represented by 33000, a lb. and foot being the units employed, and the power being exerted for one minute of time. Thus a horse in one minute is supposed to lift 33000 lbs. through 1 foot, or 3300 lbs. through 10 feet, or 330 lbs. through 100 feet, and so on. A machine is then said to be of so many horse-powers, whence the *work done* by it in any time can be calculated. Horse power.

FRICTION.

.. Hitherto the surfaces of bodies in contact have been considered *smooth*, and exerting on each other no pressure except in a normal direction. In nature, however, all surfaces are more or less rough, and when one surface is pressing or moving upon another a force is called into play which acts in a direction contrary to that of the motion, or to that in which motion would occur if the surfaces were smooth. This force is called Friction. Friction.

In machines, when a power is supporting a given weight, the magnitude of the power, determined on the supposition of the smoothness of the machine, may be increased beyond this value without disturbing the equilibrium, until it is great enough to overcome the friction together with the weight; and on the other hand, may be diminished till it is so small as with the aid of friction just to prevent the weight overcoming it. So also, with a given power, the weight may be increased or diminished within certain limits without disturbing the equilibrium. Generally, when the power is on the point of raising the weight, friction acts to the disadvantage of the power; but, when the power is just preventing the weight from descending, friction acts advantageously. When the equilibrium of a system depends on position, this position may with the aid of friction be varied within certain limits of the position determined on the supposition of smoothness, and the equilibrium be still maintained. Effect of in machines.

95. The motion of one surface upon another may be of the nature of *sliding* or *rolling*, or both these. The former will be the case when two plane surfaces are in contact, and the laws of the friction in this case (denominated *sliding friction*) have been determined by experiment, the two surfaces, however, only *tending* to slide and not in actual motion. They are,—

Sliding Friction.

Laws of.

I. Between plane surfaces of given substances, the amount of friction is independent of the extent of area in contact, and depends only on the mutual pressure between them.

II. The amount of friction is, for the same two substances, proportional to this normal pressure.

Hence, by the second of these laws, if F be the friction, and R the normal pressure, $\frac{F}{R}$ is a constant quantity for two given substances. It is called the *coefficient of friction* for these substances, and may be determined experimentally as follows:—

Coefficient of.

Found by experiment.

Fig. 20.

96. Let one of the substances form an inclined plane (fig. 20) and a block of the other, of known weight W, and having a plane base, be placed upon it; and, by varying the inclination of the plane, let that inclination (a) be found at which W is just on the point of sliding down the plane.

Then F acts upwards along the plane, and we have (§ 72.)

$$F = W \sin a$$

$$R = W \cos a.$$

Therefore the coefficient of friction $\left(\frac{F}{R}\right) = \frac{\sin a}{\cos a} = \tan a$

The values of this coefficient for various substances have been found by experiment.

AN

ELEMENTARY TREATISE

ON

MECHANICS;

DESIGNED AS A TEXT-BOOK FOR THE UNIVERSITY EXAMINATIONS
FOR THE ORDINARY DEGREE OF B. A.

PART II.

DYNAMICS OF A PARTICLE.

BY

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PREFACE TO FIRST EDITION.

The arrangement of this elementary work differs from that of most of the recent English writers on the subject, and is in the main the same as that employed by Professor Sandeman in his "Treatise on the motion of a Particle." Adopting in full the principles and method of that admirable treatise, I have attempted little more than to translate out of the language of the Calculus into ordinary algebra the investigations there given of the simpler cases of particle-motion.

For the reason stated in PART I., I have not added any *examples*, and have endeavored to be as concise as possible in any explanations or illustrations that have appeared necessary.

UNIVERSITY COLLEGE,
TORONTO,
April 1, 1858.

CHAPTER I.

THE MOTION OF A PARTICLE GEOMETRICALLY CONSIDERED.

1. When the distance between two particles changes continuously during an interval of time, they are relatively in *motion*. Motion of a point

The position, and consequently the motion, of one particle can only be conceived in relation to other particles, but it is convenient to speak of a particle *absolutely* as being at rest or in motion, reference being made to ourselves or to some points in known relation to ourselves, considering these as *fixed*, and referring all motion and change of motion to the particle itself.

By a particle is here to be understood only a geometrical point.

Uniform motion.

2. When a particle is moving in a fixed straight line, its motion is measured by the change of its distance from a fixed point in this line, and the rate of this change of distance at any instant of *time* is called the *velocity* of the particle at that instant. In a straight line. Velocity.

The change of distance in any time is here the linear space described by the particle in that time. If equal spaces are described in equal times, the change of distance in any given time is always the same, and the rate of this change, or the *velocity*, is said to be *uniform*, and is measured by the space described in a given time. Uniform, how measured.

Taking a foot and a second as the units of linear space and time, the velocity v of a particle moving uniformly will be measured by the number of feet described in one second.

Space described in any time.

The space described in 1 second being v, that in 2 seconds will be $2v$; in 3 seconds, $3v$; and, generally, in t seconds tv: hence, if s be the space described in time t, with a uniform velocity v,

$$s = vt.$$

Apparently this formula is proved only for the case where t is a whole number of seconds; but, if t be fractional, we can always assume a unit of time such that the interval of time expressed by t shall contain a whole number of these units, and the formula can then be shewn to apply. Thus let n be a whole number such that nt is also a whole number T; and let $\frac{1}{n}$th of a second be taken as the unit, and V be the velocity referred to this unit. Then the time t being expressed in this unit by a whole number T, we have $s = T.\,V = nt\,V = vt$; for v being the space in one second or n units, is n times the space in one unit, that is, $= n\,V$. Hence the formula is general.

Velocity, + and —

Distance of a moving point from a fixed point in its line of motion after any time.

3. Assuming some fixed point in the line of motion, if a be the distance of the particle from it at one instant, and s be the distance, estimated in the same direction, after the time t during which the particle has been moving uniformly with the velocity v, we shall have $s = a + vt$, or $s = a - vt$, according as the particle has been moving in the direction towards which s has been estimated *positively*, or in the opposite direction. Both these cases can be included in one formula by indicating oppositeness of direction of velocity by the algebraic signs + and —. Thus, fixing on one direction from the fixed point towards which when measured the distances are to be considered *positive*, a velocity in this direction will be *positive*, and in the opposite direction, *negative*.

Hence, if a particle move during successive intervals of time with different uniform velocities, and a be the distance from the fixed point at the beginning of the time, s its distance at the end, then

$$s = a + \Sigma\,(vt)$$

where Σ denotes the algebraic sum of all the products corresponding to that within the brackets; and the particle will be on one or the other side of the fixed point at the end of the time according as s comes out from this expression *positive* or *negative*.

The whole *space described* will, however, be the numerical sum of these products, disregarding algebraic signs.

Lemma. Distance from starting point after a continuous series of uniform motions in the same line.

4. When a particle moves from a fixed point in a straight line with different velocities during successive equal intervals of time, each velocity continuing uniform throughout its interval, the distance of the particle from the point at the end of the time is the product of the time by the arithmetic mean of all the velocities.

By the *arithmetic mean* of a number of quantities is meant their algebraic sum divided by the number of them.

For let t be the whole time; $\frac{t}{n}$ the duration of each interval; $v_1, v_2, v_3, \ldots$ the successive velocities during the first, second, third ... intervals. Then the required distance will be the algebraic sum of the spaces described with these velocities; that is, by § 2;

$$v_1 \cdot \frac{t}{n} + v_2 \cdot \frac{t}{n} + v_3 \cdot \frac{t}{n} + \ldots\ldots; \text{ or,}$$

is the product of the mean velocity and the time.

$$\frac{v_1 + v_2 + v_3 + \ldots\ldots}{n} \cdot t \quad \text{Q. E. D.}$$

The case of any of the velocities being in the opposite direction (and therefore accounted negative) is here included; the resulting sign of the algebraic sum determining on which side of the fixed point the particle is at the end of the time.

Accelerated motion.

Varying velocity, how measured.

5. When the velocity is not uniform, but changes during the motion, the velocity of the particle at any instant is measured by the space which it would describe in a unit of time, if it were to move uniformly during that unit with this velocity.

Acceleration

The rate of change of velocity at any instant (provided it be continuous) is called the *acceleration.*

Uniform, how measured.

If the change of velocity in a given time be always the same throughout the motion, the acceleration is said to be *uniform*, and it is measured by this change of velocity in a given time.

The change of velocity may be either an increase or decrease, and in the latter case the *acceleration* is in effect a *retardation*. The use of both terms is, however, rendered unnecessary by introducing the algebraic signs + and —; for a decrease is algebraically a *negative* increase, and thus a retardation is a negative acceleration; and when we speak of velocity being increased, added, or generated, we also include the case of velocity being diminished, subtracted, or destroyed.

The velocity after any period of uniform acceleration

Taking a second as the unit of time, the *acceleration* f, when the motion is *uniformly* accelerated, is the change of velocity in one second. Then $2f$ is the change in 2 seconds, $3f$ in 3; and generally tf in t seconds.

Hence, if u be the velocity at the beginning of the time t, and v be the velocity at the end of this time, we have

$$v - u = ft, \text{ or}$$
$$v = u + ft.$$

From rest.

6. If the particle started from rest at the beginning of the time, that is, if $u = 0$ then we have

$$v = ft.$$

Retardation

7. If the motion be uniformly retarded, f is to be taken negatively, and we have

$$v = u - ft.$$

The particle will be reduced to rest when $u - ft = 0$, or in a time $\frac{u}{f}$; and after this, the velocity will be accelerated in the opposite direction and by the same steps in a reverse order; till after a time $\frac{2u}{f}$, its value will be the same as at starting.

Acceleration uniform: to find the space described and the place of the particle after any time.

8. *When a particle moves with a uniformly accelerated motion from a fixed point in a straight line, to find the distance from the point after any interval of time.*

Let f be the uniform acceleration of the particle's motion; u be its velocity when at the fixed point; s the required distance from this point after a time t.

Let t be divided into n equal intervals. Then, by § 5, the velocities at the beginnings of these intervals are,

$$u,\ u+f\frac{t}{n},\ u+2f\frac{t}{n},\ \ldots\ldots,\ u+\overline{n-1}\ f\frac{t}{n};$$

and the mean of these* is $u+\frac{\overline{n-1}}{2}f\frac{t}{n}$.

Hence, by § 4, if the particle moved uniformly during each interval with the velocity at the beginning thereof, the distance required would be

$$ut+\frac{n-1}{2}f\frac{t^2}{n},\text{ or}$$

$$ut+\frac{1}{2}ft^2\left(1-\frac{1}{n}\right).$$

Similarly, if the particle moved uniformly during each interval with the velocity at the end thereof, the distance required would be

$$ut+\frac{1}{2}ft^2\left(1+\frac{1}{n}\right).$$

Between these two values the actual distance s always lies; but if we increase indefinitely the number of the assumed intervals, and diminish the duration of each, $\frac{1}{n}$ becomes indefinitely small, and each of the above quantities approaches to the same limit, which must therefore be the value of s. Hence,

$$s=ut+\frac{1}{2}ft^2.$$

Cor. 1. If the particle start from rest, then $u=0$, and we have Motion from rest.

$$s=\frac{1}{2}ft^2.$$

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* If a, l be the first and last of a series of n quantities in arithmetic progression, their sum $s=\frac{a+l}{2}.\ n$.

Hence, the mean of them $\frac{s}{n}=\frac{a+l}{2}$, or the mean of the first and last.

Motion retarded.

COR. 2. When f is positive, the velocity is continually increased, and s is the space described by the particle in the time t. But if f be negative, we have

$$s = ut - \frac{1}{2}ft^2;$$

and the distance of the particle increases till the time $\frac{u}{f}$, when it is momentarily at rest, the space described being $\frac{u^2}{2f}$. After this the particle moves back by the same stages in reverse order its distance diminishing till the time $\frac{2u}{f}$, when it is again at its starting point. It then moves to the other side of the point, s becoming negative and being now given numerically by the formula $\frac{1}{2}ft^2 - ut$, and the whole space described in the time t being $\frac{u^2}{f} + \frac{1}{2}ft^2 - ut$.

Another investigation o the sam Newton's

Fig. 1.

9. The following is another investigation of the above proposition, after Newton's manner. Draw any line AK representing on any scale the number expressing the time t, and divide AK into equal parts in the points $B, C, D,$...... Draw Aa perpendicular to AK, and take its magnitude on the same scale to represent the number expressing u the initial velocity. In the same way take Kk' to represent $u + ft$, the velocity at the end of the time. Draw ak parallel to AK, and at each of the points B, C, D..., draw perpendiculars to AK, meeting ak' in b', c', d'..., and complete the parallelograms in the figure. Then, since kk' represents ft, which is the change of velocity in the time AK; by similar triangles, dd' will represent the corresponding change in the time AD, and Dd' will represent the velocity at the end of this time, and similarly for each of the lines Bb', Cc',...... .

Now, if the particle moved *uniformly* during any interval as CD with the velocity Cc', which it has at the beginning of this interval, the space described (§ 2) would be represented

numerically by the area of the inner parallelogram Cd; so also if it moved *uniformly* during CD with the velocity Dd', which it has at the end of this interval, the space described would be represented by the area of the outer parallelogram Cd'. If, therefore, the particle moved uniformly throughout each interval on the former supposition, the whole space described would be the sum of the inner parallelograms; and if on the latter supposition, it would be the sum of the outer parallelograms; and the space (s) actually described lies numerically between the spaces described on these two suppositions. But as the number of intervals is increased, and the magnitude of each diminished, the two series of parallelograms both approach nearer and nearer to the quadrilateral area $AKk'a$, and this must therefore be the value of s. Hence s is represented by the parallelogram Ak and triangle akk', that is numerically by

$$Kk \times AK + \frac{1}{2}\, kk' \times ak, \text{ and, therefore,}$$

$$s = ut + \frac{1}{2} ft^2.$$

COR. 1. If the particle were at rest at the beginning of the time, that is, if $u = 0$, the line ak coincides with AK, and the space described is represented by the area of the triangle akk'. Hence, From rest Fig. 1

$$s = \frac{1}{2} ft^2.$$

COR. 2. If the velocity be retarded instead of accelerated, the figure will take another form. At first, the space described and the distance from the initial point at the time AK will each be represented by the area of $AKk'a$. At the time AL, the velocity will be destroyed and the particle momentarily at rest, the space described and the distance from the initial point being represented by the triangle ALa. Afterwards the particle returns towards its initial point, and the whole space described in the time AM will be represented by the sum of the areas of the triangles ALa, LMm', but the distance of the particle from the initial point will be represented by the difference of these two triangles, and at the time $AN\,(= 2\,AL)$ Motion retarded. Fig. 2.

this distance vanishes, and the particle is again at the initial point, the whole space described being represented by twice the triangle ALa. Afterwards the particle passes to the other side of the initial point, and its distance from it at the time AP is represented by the area $Nn'p'P$, while the whole space that has been described in this time is represented by the sum of the triangles ALa and LPp', that is, by twice the triangle ALa and the quadrilateral $Nn'p'P$. This result is identical with that in § 8, Cor. 3.

Motion from rest with uniform acceleration.

10. When the particle moves from rest and its motion is uniformly accelerated, we have seen that the velocity and space described at any time from the beginning of motion are given by the formulas,

$$v = ft; \; s = \frac{1}{2} ft^2,$$

and these are sufficient to determine all the circumstances of the motion in any case.

When any two of the quantities f, v, s, t, are given, the remaining two can be found from the above equations. The following cases may be noticed:

11. Given the acceleration and space described, to find the velocity acquired.

$$\text{Here } s = \frac{1}{2} ft^2 = \frac{1}{2} f\left(\frac{v}{f}\right)^2, \text{ and, therefore,}$$

$$v^2 = 2fs;$$

and conversely, to find the space through which the particle must move to acquire a given velocity, we have

$$s = \frac{v^2}{2f}.$$

12. The equation $s = \frac{1}{2} ft^2$ becomes, by putting v for ft, $s = \frac{1}{2} vt$. Hence the space described in acquiring any velocity is half the space which would be described with that velocity continued uniform through the same time.

13. Putting $t = 1$, we have $s = \frac{1}{2} f$, or $f = 2s$. Hence, twice the space described in the first second from rest measures the acceleration.

14. *The spaces described from rest in successive equal intervals of time, are as the odd numbers*, 1, 3, 5, 7, ...

For, taking any interval as the unit of time, let F be the acceleration referred to it.

Then the space described in $\overline{n-1}$ intervals from rest is $\frac{1}{2} F (n-1)^2$, and the space described in n intervals from rest is $\frac{1}{2} F n^2$.

The difference between these is the space described in the n^{th} interval, and $= Fn - \frac{1}{2} F = \frac{1}{2} F (2n - 1)$.

Giving to n the successive values 1, 2, 3, ... this becomes $\frac{1}{2} F \cdot 1$, $\frac{1}{2} F \cdot 3$, $\frac{1}{2} F \cdot 5$, ... which was to be proved.

15. The initial velocity being u, and this being uniformly accelerated during the time t, the velocity v at the end of this time and the distance s of the particle from its initial point, is given by the equations Circumstances of motion when the particle was not at rest at the commencement of the period.

$$v = u + ft; \; s = ut + \frac{1}{2} ft^2,$$

and these are sufficient to determine all the circumstances of the motion in any case.

When any three of the quantities u, f, t, v, s, are given, the remaining two can be found from the above equations.

The following cases may be noted:

16. We have
$$s = ut + \frac{1}{2} ft^2$$
$$= \frac{1}{2} t (2u + ft)$$

$$= \frac{1}{2} t (u + u + ft)$$
$$= \frac{1}{2} t (u + v)$$

or, the distance is that which would be described in the same time with a uniform velocity equal to the mean of the initial and terminal velocities.

This result might at once have been inferred from § 8.

17. Given the initial velocity, the acceleration and the distance, to find the velocity acquired.

Here u, f, s are given to find v, and t must be eliminated from the two equations.

$$v = u + ft, \; s = ut + \frac{1}{2} ft^2.$$

Squaring the first, we have

$$v^2 = u^2 + 2uft + f^2 t^2$$
$$= u^2 + 2f (ut + \frac{1}{2} ft^2)$$
$$= u^2 + 2fs.$$

If the velocity were retarded, we should have

$$v^2 = u^2 - 2fs.$$

Cor. This result might have been obtained without finding the second equation, for we have directly, from § 5,

$$v - u = ft,$$

and from § 16 or 8,

$$\frac{1}{2} t (v + u) = s;$$

multiplying these equalities, we have

$$v^2 - u^2 = 2fs.$$

The following geometrical proof may also be noticed:

Let B be the initial point, where the velocity is u; BC the space described (s) when the velocity is v.

Let A be the point from which the particle, proceeding from rest under the same acceleration, would acquire the velocity u at B. Then (§ 11).

$$u^2 = 2\,f.\;A\,B.$$

Also, since the whole motion may be taken to proceed from rest at A, we have (§ 11)

$$\begin{aligned} v^2 &= 2\,f.\;A\,C \\ &= 2\,f.\,(A\,B + A\,C) \\ &= 2\,f.\;A\,B + 2\,f.\;A\,C \\ &= u^2 + 2\,fs. \end{aligned}$$

By a proper adaptation of the figure, this proof may be extended to all the cases included in the algebraic formula.

18. Given the initial velocity and the acceleration, to find the time when the particle will be at a given distance from the initial point.

Here u, f, s are given to find t.

Solving as a quadratic in t the equation $s = u\,t + \frac{1}{2}ft^2$, we have

$$t = \frac{-u \pm \sqrt{u^2 + 2\,fs}.}{f}$$

The significance of the double sign is here note-worthy.

If f be positive, or the velocity be numerically accelerated, one of the values of t is positive, and the other negative. The former is the solution required, but the latter can be interpreted thus: Suppose A the initial point, AP the distance s, and the velocity u at A to be in the direction AP. Then the positive value of t in the above gives the time of moving from A to P; the negative value gives the time that would have elapsed if the particle had moved from P towards A, with a retarded motion, passed through A to the other side of it, been reduced to rest and again returned to A.

If f be negative, then, writing $-f$ for f, the values of t become

$$\frac{u \mp \sqrt{u^2 - 2\,fs}.}{f}$$

If then $u^2 > 2fs$, both values of t are real and positive, and the particle will twice be at the same distance from the initial point, once during the recess from and again during the return towards it.

If $u^2 = 2fs$, the two values become the same, and the distance in question is that where the particle momentarily comes to rest.

If $u^2 < 2fs$, both values of t are imaginary, and the particle can never reach that distance.

If, however, s be negative, both values are real, and one positive, the other negative, the latter referring to a time previous to the epoch from which we are reckoning, when the particle, if it had been moving towards the initial point from the negative side, would have been at the assumed distance.

Component Velocities.

Composition of velocities.

Fig. 3.

19. The position and motion of a particle moving uniformly in a straight line have been determined by the distance of the particle from a fixed point in the line, and by the change of this distance in a given time. Its position, however, might have been defined by its distances from two fixed lines, measured parallel to these lines. Thus: let Ox, Oy be two fixed lines, B the place of a particle moving in the line ABC, and A a fixed point in this line, the distance from which determines the place of the particle. Let C be the place at which the particle would arrive after any time if it moved uniformly with the velocity it had at B, and complete the figure by drawing lines parallel to Ox, Oy.

The position of B is known when BP, BQ, its distances from these fixed lines, are given; and CE, CD, or their equals, BD, BE, would be the changes of these distances if the particle arrived at C by moving uniformly.

Now BC, which would be the change of distance in a given time from the fixed point A, measures the velocity of the par-

ticle : and BD, BE are *always* proportional to BC, and therefore measure what we may call the *component velocities* of the particle in the directions of the fixed lines. Hence,

Parallelogram of component velocities

If a straight line be taken to represent in magnitude and direction the velocity of a particle, the adjacent sides of any parallelogram constructed on this line as diagonal will represent the COMPONENT VELOCITIES *in the directions of those sides.*

Conversely.

If the COMPONENT VELOCITIES *in two directions be given, the actual velocity will be found in magnitude and direction by drawing the diagonal of the parallelogram of which the components form adjacent sides.*

These two statements constitute the "parallelogram of component velocities."

Velocity resolved in any direction.

20. When the two components are in perpendicular directions, it will be convenient to call them the *resolved parts* of the velocity in these directions; and the rule for finding these resolved parts will be the same as that for the resolved parts of a Force (STATICS, § 21), namely :

Rule for.

To find the resolved part of a velocity in any direction, *multiply it by the cosine of the angle between this direction and that of the velocity ;* and to find the resolved part perpendicular to this direction, *multiply by the sine of the aforesaid angle.*

CHAPTER II.

THE MOTION OF A MATERIAL PARTICLE ACTED ON BY UNIFORM FORCES.

Application of preceding results to the actual motions of material particles.

21. In the foregoing chapter the geometrical conditions of the motion of a point have been examined. It now remains to exhibit the connection of these results with the actual motions of material particles, and the relation between these motions and the forces acting on the particles, and this investigation constitutes the science of *Dynamics*.

Experimental Laws.

For this purpose it is necessary to appeal to experiment and observation, and it appears that all the phenomena of the motions of material particles can be referred to three elementary principles or laws, which are commonly known as "Newton's Laws of Motion." These laws, from their nature, are incapable of being demonstrated by direct experiment, for it is impossible to make experiments under the precise circumstances conditioned by the Laws, and which would not involve other phenomena besides those which it is desired to test. Direct experiments may, however, afford a presumption in favor of these laws by showing that the more nearly do the circumstances of the experiment approach to the exact conditions required, the more nearly are the results of the experiment in accord with those indicated by the Laws; and also that whenever a discrepancy is found between these results, there can always be traced some disturbing cause which ought to have been excluded by the conditions postulated.

The ultimate ground on which these and all other laws in Natural Philosophy rest, is the entire and universal concordance of the results of experiment or observation with those calculated on the assumption of the truth of the laws.

22. Although the motion of a particle and the forces acting on it can only be conceived in relation to other particles, it is convenient to speak *absolutely* of a particle as being at rest or in motion, reference being made to ourselves or to some space in a known relation to ourselves which we consider *fixed*, and then to regard the phenomena exhibited by the particle as due to forces acting only on itself, these forces being defined by the measures of them already employed in Statics.

Newton's three Laws of Motion.

23. FIRST LAW OF MOTION.*

First Law.

A material particle, when not acted on by any force, if it rest, will so remain; and, if in motion, will move in a straight line with uniform velocity.

No forces acting,

The first part of this law has been already assumed (Statics § 2) as the basis of our conception of a force. Experience shows that whenever a quiescent body is set in motion, we can trace the action of some cause external to the body; thus, when a body is suffered to drop to the earth, we assign its motion to a pressure exerted on it due to the earth itself, and which would have no existence if the earth did not exist. Also, there seems no reason why a particle, apart from any external force, should begin to move in one direction rather than another.

the particle either remains at rest, or

Again, when a particle is in motion there seems no reason why it should change the direction of its motion in one way rather than another, unless some force be acting upon it to determine such change; and in all cases of any such change, we can always trace the action of some external force; as, for instance, when a stone is projected from the earth in any direction, the deflection of its motion from a straight line is produced by the aforesaid pressure due to the earth, which we know is always acting vertically downwards. If this pressure be counteracted by projecting the stone horizontally along a

moves in a straight line.

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* LEX. I. *Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus a viribus impressis cogitur statum illum mutare.*—PRINC. *Leg. Mot.*

fixed plane, the path approaches to a straight line, with only such deviations as may be accounted for by friction or irregularities in the plane, or from the stone not being small enough to be considered a particle.

with uniform velocity.

Galileo.

So also with regard to the velocity of the particle, it does not seem possible to conceive any way in which its velocity could increase or decrease unless by the action of some external cause and in actual cases of variation of velocity we can always trace the existence of such causes. Thus when a stone is thrown horizontally along the ground, it gradually loses its velocity and stops, but here the friction of the ground and the resistance of the air act as retarding causes, and we see that in proportion as the surface on which the stone moves is smoother, as on a sheet of ice, the longer and more uniform does the motion continue.

This law is sometimes termed the *Law of Inertia*, being understood to express that a material particle is *inert*, and has no tendency of itself to change its state of rest or uniform motion.

24. It follows that the motion of a material particle when not acted on by any forces, or acted on only by forces which counterbalance, is determined by the formula of uniform motion, $s = vt$, investigated in § 2.

Uniform forces acting on a particle.

25. We now proceed to consider the motion of a particle acted on by any uniform forces, of which the following are the observed laws :—

(1.) *When a uniform force acts continuously upon a particle in the line of its motion, the velocity is uniformly accelerated.*

A single force in the line of motion.

The investigations of § 5 *et seq.*, therefore, apply to this case, (noticing also that *retardation* is included in the term *acceleration*), and we can compare the results there calculated with those of experiment. Thus when a body is permitted to drop freely to the earth, or is projected vertically downwards or upwards with any assigned velocity, its path is a vertical line, and the force acting on it is its weight which always acts ver-

tically, and (for not great heights above the surface) is sensibly uniform. Here then the required conditions are fulfilled, and the result of experiment, when due allowance is made for the resistance of the air, is that the motion is uniformly accelerated, the amount of this acceleration being about 32.2 feet a second, but varying slightly for different latitudes and elevations above the sea-level. This acceleration is usually denoted by g.

(2.) *When several uniform forces are acting simultaneously in the line of motion, the resulting acceleration is the algebraic sum of the accelerations which would be produced by each force acting separately.* Any forces in the line of motion

Hence it follows that n equal forces acting simultaneously on a particle in its line of motion will produce n times the acceleration which one of the forces alone would produce on the same particle; and, consequently, the acceleration produced in a given particle is proportional to the magnitude of the force acting.

It will be hereafter shown how this may be tested by comparing the accelerations of a particle down inclined planes of different inclinations.

Hence also the change of velocity in a given time is proportional to the magnitude of the Force, the particle acted on being the same.

(3.) When a moving particle is acted on continuously by a uniform force which acts always in the same direction and obliquely to the direction of the particle's motion, its velocity after any time is found to have for components—first, the original velocity, unaltered, in its own direction—second, a velocity in the direction of the Force, the magnitude of which is the same as if the Force had acted on the particle originally at rest. So that the velocity and direction of the motion may be found at any time by calculating the velocity which would be produced by the Force acting for that time on the particle originally at rest, and then compounding this with the original velocity according to the principle of the "parallelogram of velocities." A single force oblique to the direction of motion.

Or, this may be expressed more simply thus: if we resolve the original velocity of the particle into two components, one in direction of the force, the other perpendicular to it, the latter remains unaltered and the former is changed by the Force precisely as if it alone were the actual velocity of the particle. So that

The change of velocity produced by the force in a given time is in direction of the force and is proportional to it in magnitude.

In this case the path of the particle is no longer a straight line, but a curve, the tangent to which at any point is the direction of the particle's motion there.

In other words, the above expresses that *the dynamical effect of a force on a particle is wholly independent of any motion which the particle may have, and is the same as if it were exerted on the particle originally at rest.*

Thus, the vertical descent of a body let fall from the masthead of a ship in motion is precisely the same in all its circumstances as if the ship were at rest. The principle can also be tested by comparing the results of calculation with observations on the motion of a body projected obliquely to the horizon and acted on by gravity, due allowance being made for the resistance of the air.

Any forces acting in any direction on a particle either originally at rest

(4). When several Forces act simultaneously, retaining always the same magnitudes and directions, on a particle originally at rest, the motion is uniformly accelerated in the direction of the Resultant of the Forces, and the acceleration is that due to this Resultant acting singly.

Also the velocity generated after any time, being that due to this Resultant, is also that which is compounded of the velocities due to the Forces acting singly on the particle from rest.

or in motion.

Also if the particle be in motion when the Forces begin to act, its velocity and direction of motion after any time will

be determined by compounding its original velocity with the velocity due to the Resultant of the Forces, or with all those due to the Forces separately. Or,

When forces act on the same particle under any circumstances provided each force be uniform and always preserve the same direction, the change of velocity in a given time due to each force is in direction of that force, and is proportional to it in magnitude.

(5.) It follows from the preceding that if f be the acceleration due to a force P acting on a certain particle, then the ratio $P : f$ is invariable for this particle. This ratio is found, however, to be different in different particles, and we thus discover a quality which distinguishes one particle from another, of which this ratio will serve as a measure. The name of *mass* is given to it, and one particle has the same *mass* as another when the same force produces in each the same acceleration. The unit of mass is arbitrary and it is not necessary to fix it, but we shall take as the measure of *mass* the above ratio of the numbers expressing a Force and the acceleration it produces on the particle. Thus, if m be the *mass* of a particle, and P, f as above,

For the same particle the acceleration is proportional to the force.

Mass defined,

and measured.

$$\frac{P}{f} = m$$

It has been mentioned that the acceleration produced by gravity (g) is the same for all bodies at the same place on the Earth's surface. Hence, if W be the weight of a body, m its mass, we have

$$\frac{W}{g} = m, \text{ and}$$

$$W = m\,g.$$

Hence, for a given place, the weights of bodies may be taken to measure their masses.

The fact above stated (namely, that the acceleration of gravity is the same for all bodies at the same place) is apparently contradicted

Lucretius. by the different times occupied by different bodies in falling from the same height to the earth; but this is due to the different resistances of the air, as is shown by trial in an exhausted receiver, where the feather and the guinea are seen to fall precisely in the same time.

(6.) Since $P = m f$, and f is proportional to the change of velocity in a given time (§ 5), it follows that P is proportional to the product of the mass and the change of velocity in a given time.

Momentum. The product of the mass and the velocity (that is, of the numbers expressing these,) is called the *momentum* of the particle, and the preceding results can now all be combined in one statement, which constitutes

THE SECOND LAW OF MOTION.*

Second Law. *When uniform Forces act continuously for a given time on material particles, each produces in its own direction a change of momentum proportional to itself in magnitude.*

Impulsive Forces. 26. The forces hitherto treated of have been of such a nature as only to produce finite changes in the motion of a particle by acting on it for a finite time. There is, however, a certain class of forces, such as those manifested in explosions or the collision of bodies, which produce finite effects in changing the velocity or momentum *instantaneously*. Such forces are called *impulsive*, and must be carefully distinguished from forces of the former class, with which they do not in any way admit of comparison. These impulsive forces are measured by the momentum which each would instantaneously communicate to a particle at rest, and the second Law of Motion applies to them, stated under the form:

Second Law applied to. *When impulsive Forces act on material particles, each produces in its own direction an instantaneous change of momentum proportional to itself in magnitude.*

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* Lex. II.—*Mutationem motus proportionalem esse vi motrici impressæ, et fieri secundum lineam rectam qua vis illa imprimitur.*—Princ. Leg. Mot.

Thus the motion of a particle when acted on by simultaneous impulses will be determined by calculating the velocity instantaneously generated by each in its own direction, and compounding these with the original velocity of the particle according to the parallelogram of velocities. For instance, if a particle at rest be acted on by two impulses which, separately communicated, would give the particle respectively such velocities as would cause it to describe uniformly the sides AB, AC of a parallelogram $BACD$, the particle will acquire from the impulses simultaneously communicated a velocity which will cause it to describe uniformly the diagonal AD in that time.

Parallelogram of velocities. Aristotle.

APPLICATIONS AND TESTS OF THE SECOND LAW OF MOTION.

Vertical motion by gravity. Galileo.

27. ***The vertical motion of a particle under the action of gravity.***

$g = 32{\cdot}2$

The acceleration of gravity (g) has already been stated to be about 32·2 feet per second, and to be sensibly the same for all bodies in the same latitude and at nearly the same height above the sea-level.

From rest

Hence, applying the formulas in § 10, we have, when the particle moves from rest,

$$v = gt = 32{\cdot}2 \times t;\ s = \frac{1}{2}gt^2 = 16{\cdot}1 \times t^2.$$

Thus the spaces described from rest after the lapse of 1, 2, 3, ... seconds, are 16·1, 64·4, 144·9, ... feet; and the velocities acquired are 32·2, 64·4, 96·6, ... feet per second.

Projected down

If the body do not fall from rest, but be projected downwards with a velocity u, then we have § 15

$$v = u + gt,\ s = ut + \frac{1}{2}gt^2.$$

or up.

Or if it be projected vertically upwards with a velocity u, then the velocity and distance from the point of projection at the time t are given by

$$v = u - gt,\ s = ut - \frac{1}{2}gt^2;$$

and the particle is brought momentarily to rest after having ascended a height $\frac{u^2}{2g}$ in the time $\frac{u}{g}$, and then descends again by the same steps in reverse order, reaching its point of projection in the time $\frac{2u}{g}$.

The resistance of the air and the rapidity of the motion render it difficult to test these results directly by experiment.

Motion down an inclined plane.

Galileo.

28. *Motion down a smooth inclined plane.*

Let a be the inclination of the plane.

The particle is acted on by two forces, namely, its own weight (W) acting vertically, and the reaction of the plane in a normal direction. If we resolve W into two forces, one perpendicular to the plane, and the other ($W \sin a$) downwards along the plane, the motion estimated along the plane will be due to this latter only. Hence, f being the acceleration along the plane, we have § 26 (5),

$$\begin{aligned} f &= \text{Force} \div \text{mass of particle} \\ &= W \sin a \div \frac{W}{g} \\ &= g \sin a\,. \end{aligned}$$

And with this value of f the formulas of § 10 and 15 avail to determine fully the motion.

L. da Vinci.

Cor. For the same particle, on planes of different inclinations, the accelerations are as the sines of the inclinations, or, the length of the plane being given, as the heights; and this is the test mentioned in § 25, (2), allowance being made in performing the experiment for the resistance of the air and imperfect smoothness.

29. *The velocity acquired by moving from rest down an inclined plane, is equal to that acquired by falling freely down the height of the plane.*

For, f the acceleration down the plane is g sin a, and if v be the velocity acquired in moving down its length, § 11,

$$\begin{aligned} v^2 &= 2 f \times \text{length} \\ &= 2 g \sin a \times \text{length} \\ &= 2 g \times \text{height}; \end{aligned}$$

and this is the same as if the body fell freely down this height.

Cor. If the particle were projected down or up the plane with a velocity u, and v be the velocity after moving through any *length* of it, we should have in like manner, § 17,

$$\begin{aligned} v^2 &= u^2 \pm 2 g \sin a \times \text{length} \\ &= u^2 \pm 2 g \times \text{height}, \end{aligned}$$

which is the same as if the particle were projected vertically downwards or upwards with a velocity u, and moved freely through the corresponding *height*.

30. *The time of moving from rest at the highest point of a vertical circle down any chord (considered a smooth inclined plane) is the same as the time of falling freely from rest down the vertical diameter; and so is also the time of moving from rest down any chord to the lowest point.*

For, AB being the vertical diameter, the acceleration down the chord AC is g cos BAC, and therefore, § 10, Fig. 4.

$$(\text{time down } A\,C)^2 = \frac{2\,A\,C}{g \cos B\,A\,C} = \frac{2\,A\,B}{g},$$

which is the square of the time down $A\,B$.

So also, the acceleration down $C\,B$ is g cos $C\,B\,A$, and

$$(\text{time down } C\,B)^2 = \frac{2\,C\,B}{g \cos C\,B\,A} = \frac{2\,A\,B}{g},$$

the same as in the former case.

A particle projected in any direction and acted on by gravity.

Galileo.

81. *Motion of a projectile.*

Let a particle be projected from a point in the horizon, with a velocity v, and in a direction making an angle a with the horizon. The force acting on it being its weight which always is directed vertically downwards, the motion of the particle will be in one vertical plane.

If we resolve its velocity of projection into two: namely, $v \cos a$ horizontally, and $v \sin a$ vertically; the former continues unaltered, and the latter is retarded and accelerated by gravity precisely as if the particle had been projected vertically with this velocity. Hence, g being the acceleration by gravity, the velocity $v \sin a$ is destroyed by it in a time $\frac{v \sin a}{g}$, (§10); at this moment the particle is moving horizontally, and has reached its greatest elevation above the horizon. The velocity $v \sin a$ is again generated by gravity by the same steps in a reverse order, till on again reaching the horizon the velocity is the same in magnitude, and its direction is equally inclined to the horizon in an opposite direction, as at the point of projection. The path, therefore, consists of two equal and similar branches on each side of the greatest elevation.

Time of flight.

The whole *time of flight* is therefore $2 . \frac{v \sin a}{g}$, and during this time the horizontal distance described with the uniform velocity $v \cos a$ is (§ 2)

$$v \cos a . \; 2 . \frac{v \sin a}{g}, \text{ or}$$

$$2 \frac{v^2}{g} \sin a \cos a, \text{ or}$$

$$\frac{v^2}{g} \sin 2a. \quad \text{(Trig. § 72.)}$$

Range.

and this is called the *Range*.

The greatest elevation is the space due to the velocity $v \sin a$ for the acceleration g: that is (§ 11),

Greatest height.

$$\frac{(v \sin a)^2}{2g}$$

Again, if x be the horizontal distance of the particle from the point of projection at the time t, and y its elevation above the horizon at that time, we have (§§ 2 & 15), Place at any time.

$$x = v \cos a . t$$
$$y = v \sin a . t - \tfrac{1}{2} g t^2$$

which determine the place of the particle at any time.

The path of the particle is the curve called by geometers a parabola. In comparing these results with observation, the resistance of the air has to be taken into account; and for large bodies, or considerable velocities, these results are thereby rendered quite wide of the observed facts.

32. THIRD LAW OF MOTION.*

When one material particle acts on another, the second exerts on the first an action equal in amount and opposite in direction to that which the first exerts on it. Third law.

The actions here spoken of may be of various kinds; such as the mutual pressures between bodies in contact whether at rest or in motion; or the action of one particle on another by means of a stretched string or a rigid rod; or the action may be of the nature of attraction or repulsion; or finally of an *impulsive* character, as in cases of collision.

The measures of these actions are either their statical measures or those furnished by the second law of motion.

Sometimes the law is stated in the form:

The actions of bodies are mutual, equal, and opposite.

The law may be tested by direct experiment in various ways; such as by noting the motion of two bodies hanging freely by a string passing over a pully: by observing the motion of a magnet and piece of iron floating on water: and by observing the velocities of balls that have suffered collision.

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* LEX. III. *Actioni contrariam semper et æqualem esse reactionem: sive, corporum duorum actiones in se mutuo semper esse æquales et in partes contrarias dirigi.*—PRINC. LEG. MOT.

Motion of two weights over a pully. Newton.

33. **Two bodies connected by an inextensible string, which passes over a smooth fixed pully, descend by the action of gravity, to determine the motion.**

Let P, Q be the weights of the two bodies, P being the greater of the two.

The pully being smooth, and the weight of the string insensible, the tension of the string is, by the third Law, the same on each body.

Since the string is supposed inextensible, the downward motion of P is the same as the upward motion of Q, and we may consider them as one mass acted on in the direction of motion by the uniform pressure $P - Q$

The weight of the mass moved is $P + Q$, and the mass is therefore $\frac{P+Q}{g}$.

But the pressure divided by the mass is the acceleration (f); hence,

$$f = (P - Q) \div \frac{P+Q}{g}$$
$$= \frac{P-Q}{P+Q} g,$$

and with this value of f, the formulas of §10 & 15 apply.

Atwood's machine.

Fig. 5.

34. It has been mentioned that the velocity of a body fallingly freely is too rapid to be conveniently experimented on. In the above case of motion, however, the acceleration can be made sufficiently small, depending as it does on the difference between P and Q, to enable observations to be made with some accuracy. This is effected by the arrangement known as *Atwood's Machine,* which consists essentially of two weights connected by a thin string passing over a pully, and the disturbance caused by friction is lessened by the axle of the pully being made to rest on friction-wheels. The resistance of the air, however, still interferes with the results.

The weights are contained in two boxes A, B, and the motion of the descending one A is measured by means of a vertical rod, graduated in inches, on which a moveable stage can be fixed at any point, and the coincidence of the striking of the bottom of the box on this stage with the beat of a second's pendulum attached to the machine is employed to measure the time.

The weights used are equal pieces of brass, denominated *ems*, and it is found that the effect of friction and of the motion of the pully can be represented by supposing an additional number of these *ems* to be added to the whole weight.

Thus, the weight of the two boxes being 12, and each being loaded with 20 *ems*, in which condition they would balance, let 1 *em* be added to A, and 11 *ems* be allowed for friction and the effect of the pully.

Then the weight of the whole mass moved $(P+Q)$ is taken as 64 ems, and the moving pressure $(P-Q)$ is 1 em. Therefore, applying the formula in the preceding article,

$$f = \frac{1}{64} g.$$

On making the experiment the spaces described in 1, 2, 3, 4 seconds respectively are found to be about 3, 12, 27, 48 inches. And applying the formula $s = \frac{1}{2} ft^2$, the value of g comes out in each case 32 ft. per second.

The experiment may be varied in several ways, and we have thus a test not only of the third law, but of the uniformity of the acceleration produced by gravity, and of its constant value for bodies of different weights, § 25 (1). The machine may also be employed to test the proportionality of the acceleration to the moving pressure, § 25 (2), the mass moved remaining the same.

Thus, the boxes being loaded so as to balance, one or more *ems* in the form of long bars can be placed on A as a moving pressure, and a ring-stage can be fixed at any point of the

vertical bar which permits the box to pass freely, but removes the long *ems*. After which removal the box continues to descend uniformly with the velocity acquired (or would do so but for friction and the resistance of the air), and the space thus described in one second can be measured, giving the velocity acquired, and therefore the acceleration.

For example, let the boxes be loaded each with 20 ems; their weight 12; allowance for friction, &c., 11; and let 1 long em be added to A. Then the weight of the whole mass is 64. After the box has descended from rest through one second, let the long *em* be removed by the ring stage. Then the space described in the next second by the box moving uniformly is found to be six inches, and this is the measure of the velocity acquired in the first second, and therefore of the acceleration.

Hence, the mass being 64, and the moving pressure 1, the acceleration is 6.

Again, let the boxes be loaded each with 19 *ems*, and 3 long *ems* be put on A; and by the same method the acceleration is found to be 18.

Thus, the mass being 64, and the moving pressure 3, the acceleration is 18.

So that the acceleration is proportional to the pressure.

Again, the effect of gravity as a retarding force may be exhibited by allowing box A to acquire a certain velocity in its descent, then removing the long *ems*, so that the other box becomes the heavier, and noting the time when they are reduced momentarily to rest.

Impact and Collision.

Collision of smooth balls.

Newton.

35. Since balls are extended bodies, we cannot apply to them directly the laws for the motion of mere *particles*, but when the balls are uniform in substance, the motion of their centres will be the same as that of imaginary particles of the same masses as the balls, placed in these centres.

Direct impact of two balls.

Two balls impinge directly.

36. The impact is said to be direct when the centres of the two balls are moving in the same line. The impulsive action which occurs between the balls on collision takes place wholly in this line (the balls being smooth), and is measured (§ 26) by the instantaneous change of momentum in each ball in this direction, and, by the third law of motion, this must be the same in amount and opposite in direction on each ball. That is, the gain of momentum by one ball is the same as the loss by the other, and the (algebraic) sum of the momenta of the two balls remains unaltered by the impact; or, in other words,

Law of equality of momenta.

The algebraic sum of the momenta after impact is equal to that before the impact.

This gives one relation between the velocities after impact; but, to determine them, another relation is still required. This is furnished by the following experimental law:

Law of relative velocities.

The two balls either proceed in contact with a common velocity, or they separate in such a manner, that the magnitude of their relative velocity after impact bears to that of their relative velocity before impact a ratio which depends only on the nature of the substance of which the balls are composed.

Elasticity.

In the former case the balls are called *inelastic;* in the latter, *elastic;* and the constant ratio above spoken of is called the *elasticity* for each particular substance, and is generally denoted by the letter e. Its value for all known substances lies between 0 and 1; thus for steel it is $\frac{5}{9}$; for ivory, $\frac{8}{9}$; for glass, $\frac{15}{16}$. If e were equal to 1 for any substance, it would be *perfectly* elastic, but no such substance is known in nature. It is clear that the case of inelastic bodies is included in that of elastic ones by giving to e the value 0.

By the *relative velocity* of the balls is meant the algebraic difference of their velocities. Thus if u, v are the velocities in the same direction before impact, and the ball moving with u

overtakes the other; then $u - v$ is their relative velocity; and if u', v', are the corresponding velocities after impact, the second ball moving away from the first, then $v' - u'$ is the relative velocity, and the experimental law asserts that,

$$\frac{v' - u'}{u - v} = e.$$

Two inelastic balls impinging directly.

37. *Two inelastic balls impinge directly with given velocities, to find their velocity after impact.*

Let A, B be the masses of the two balls, and u, v their velocities estimated in the same direction; then, after impact, they proceed with a common velocity, V (suppose). The algebraic sum of the momenta before impact is

$$Au + Bv;$$

and, after impact, it is

$$(A + B)\ V.$$

Hence, by the law of equality of momenta,

$$(A + B)\ V = Au + Bv, \text{ and}$$

$$V = \frac{Au + Bv}{A + B}.$$

If the second ball were moving in an opposite direction to the first, v would be taken negative, and the direction of V will be indicated by its resulting sign.

Cor. 1. If B were at rest, then $v = 0$, and

$$V = \frac{A}{A + B}\ u.$$

Cor. 2. If the balls be brought to rest by the impact, then $V = 0$, and, therefore,

$$Au + Bv = 0,$$

$$\text{or } \frac{u}{-v} = \frac{B}{A}.$$

Or the balls must have been moving in opposite directions with velocities inversely proportional to their respective masses.

88. *Two elastic balls impinge directly with given velocities, to determine their velocities after impact.* Two elastic balls impinging directly.

Let A, B, be the masses of the two balls:

u, v, their velocities before impact, estimated in same direction,
u', v', " after " " "
e, the elasticity.

A is supposed to overtake B.

Then the sum of their momenta before impact is $Au + Bv$.
and " " after " $Au' + Bv'$.

Hence, by the law of equality of momenta,

$$Au' + Bv' = Au + Bv.$$

Again, by the law of relative velocities,

$$v' - u' = e\,(u - v).$$

From these two equations, finding u' and v', we have

$$u' = \frac{Au + Bv - Be\,(u - v)}{A + B}$$

$$v' = \frac{Au + Bv + Ae\,(u - v)}{A + B}.$$

If B were moving before impact in an opposite direction, v would be taken negative, and the directions of u', v' will be indicated by their algebraic signs.

Cor. 1. In no case can both balls be brought to rest by the impact.

Cor. 2. If the balls be perfectly elastic, or $e = 1$; and if also their masses be equal, or $A = B$; then we have Two equal and perfectly elastic balls exchange velocities.

$$u' = v, \quad v' = u,$$

and the two balls exchange velocities.

Thus, if the second were at rest, the first after impact would remain at rest, and the second would go on with the velocity of the first before impact.

Cor. 3. Hence, if a row of equal, perfectly elastic balls be ranged in contact in a straight line, and another ball, also perfectly elastic and equal to each of them, impinge in each line on the first of these balls, the impinging ball will remain at rest after the impact, and the first ball will start with the same velocity; it will then impinge on the second, communicating to it this velocity, and itself remaining at rest; the second on the third, and so on, till the last ball flies off with the velocity, all the others remaining at rest.

Again, if two such balls, moving with the same velocity, impinge on the row of balls, the first of the impinging balls will, as before, drive off the last of the row; and the second will then drive off the last but one, all the others remaining at rest. And so on for any number of impinging balls, whether greater or less than that of the balls struck, the number of balls driven off will be the same as that of the impinging balls, the others remaining at rest.

Impact on a fixed body, deduced.

Cor. 4. In the general case, suppose B to be at rest before impact, then $v = 0$, and we have

$$u' = \frac{A - Be}{A + B}\,u, \quad v' = \frac{A + Ae}{A + B}\,u.$$

Now suppose B to become indefinitely great compared with A; then the limiting values of $\frac{A - Be}{A + B}$ and $\frac{A + Ae}{A + B}$ are $-e$ and 0. Hence,

$$u' = -eu, \quad v' = 0,$$

and we have the case of a ball impinging directly on a *fixed* body, and the first equation shows that the velocity after impact is in the opposite direction to that before impact, and its magnitude is less in the ratio of e to 1.

39. *Oblique impact of smooth balls.*

Oblique impact.

Here the centres of the balls are not moving in the same line, but the impulsive action takes place (the balls being smooth) in the line joining their centres at impact. If therefore the velocity of each ball be resolved in two directions; one, along the line joining the centres; the other, perpendicular to this line; the latter resolved parts will not be altered by the impact, and the former will be altered precisely as if the

impact were direct, and the resolved velocities in this direction after impact can be calculated by the preceding investigation, and then compounded with those in the perpendicular direction by the parallelogram of velocities, thus determining fully the motion and direction of motion of each ball.

40. *Oblique impact against a smooth fixed plane.* Impact at a fixed plane.

The impulsive action exerted by the plane is in a normal direction, the plane being smooth. Hence, if we resolve the velocity of the ball in two directions; one, along the plane; the other, normal to it: the former will be unaffected by the impact, while the latter will be changed just as if the impact were direct, that is, its direction will be reversed and its magnitude diminished in the ratio of $e : 1$. Combining these velocities, the motion and direction of motion of the ball after impact are determined. Thus:

Let v be the velocity of the particle, θ the angle its direction makes with the normal to the plane at the point of impact.

Let v' be the velocity after impact, and θ' the corresponding angle. Then $v \cos \theta$, $v \sin \theta$ are the velocities respectively normal to and along the plane, and we have

$$e v \cos \theta = v' \cos \theta'$$

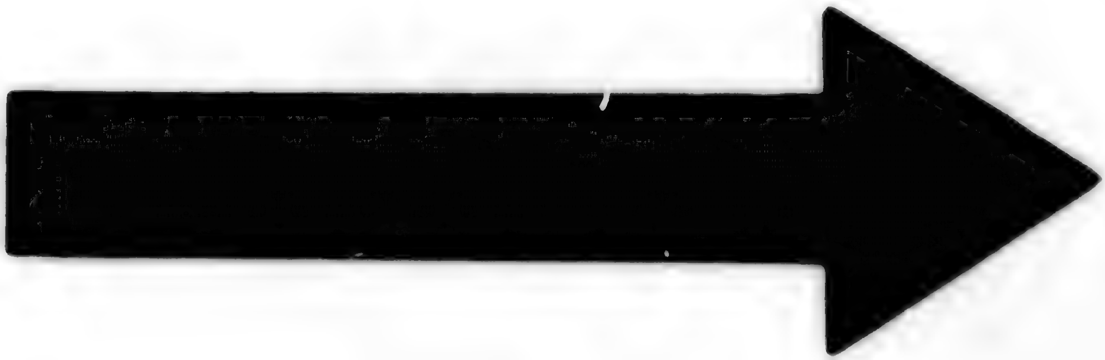
$$v \sin \theta = v' \sin \theta'$$

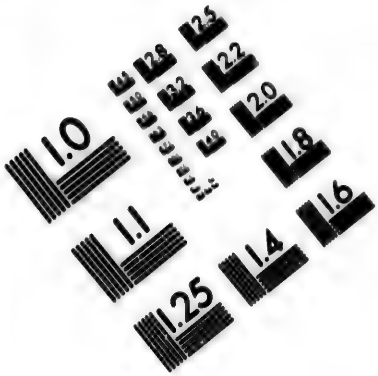
from which we obtain

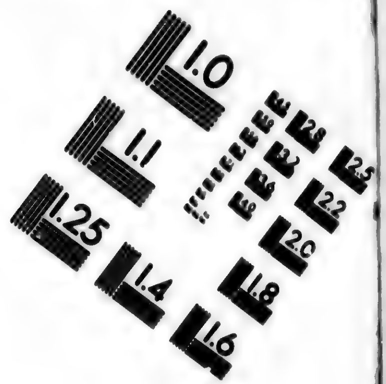
$$v' = v \sqrt{\left\{ \sin^2 \theta + e^2 \cos^2 \theta \right\}}, \text{ and}$$

$$\tan \theta' = \frac{1}{e} \tan \theta.$$

Or, the same may be done by an easy geometrical construction; thus, take AC to represent the velocity at impact, CM the normal, AM perpendicular to CM. Then AM, MC represent the component velocities along and perpendicular to the plane. Take $CN = e\ CM$, NB perpendicular to CM and equal to MA; join CB: then CB will represent in magnitude and direction the velocity after impact. Geometrical construction. Fig. 6.







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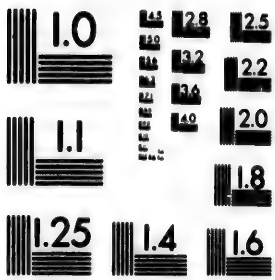

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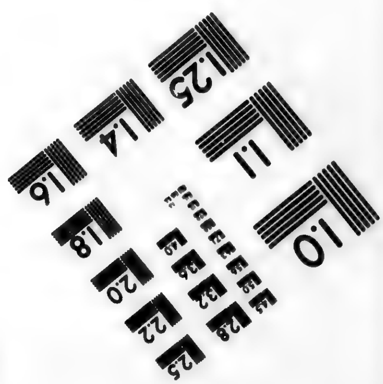

With perfect elasticity, the angles of incidence and reflexion are equal.

Cor. If the elasticity be perfect, the velocity is the same after impact as before, and its direction is equally inclined to the normal on the other side of it.

This furnishes a simple construction for finding the direction in which a particle in a given position must be projected so as after reflexion at a fixed plane (perfectly elastic) to strike a given point. The rule will be: aim at a point directly behind the plane at the same distance from it as the point required to be hit.

How to hit a required point after successive reflexions at any number of planes.

So also, if it be desired to strike a given point after reflexion at two fixed planes in succession, imagine a point at the same distance directly behind the second plane as the given point is before it, and then aim at a point directly behind the first plane at the same distance from it as this imaginary point is before it. And a similar construction serves for the same problem after successive reflexions at any number of planes.

THE END.

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Fig. 1.

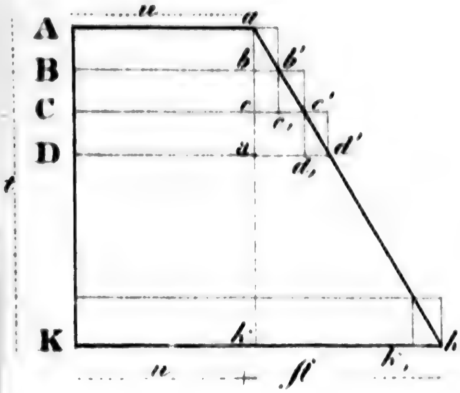


Fig. 2.

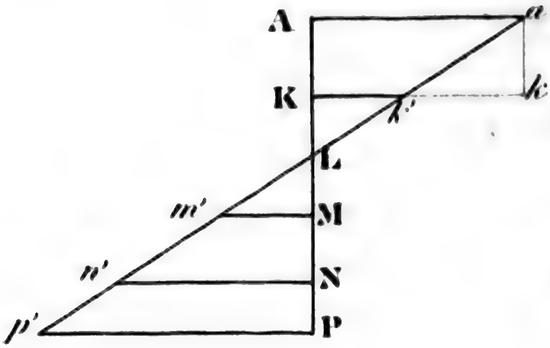


Fig. 3.

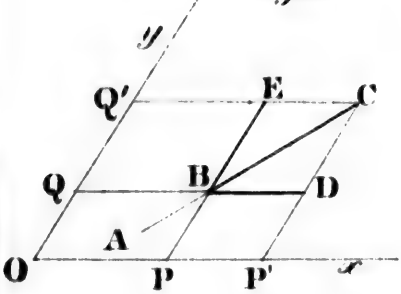


Fig. 4.

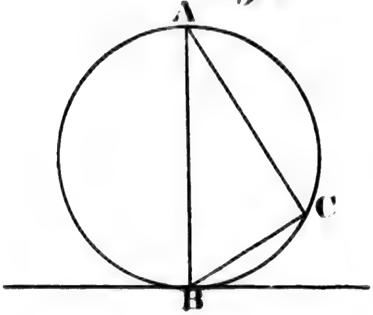


Fig. 5.

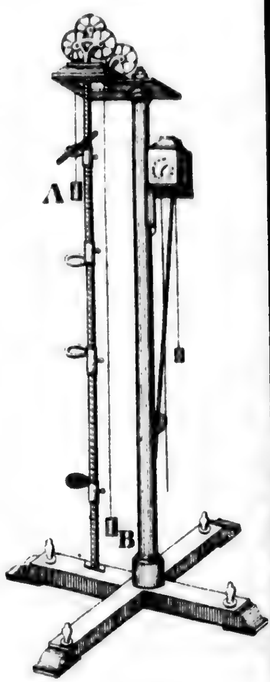
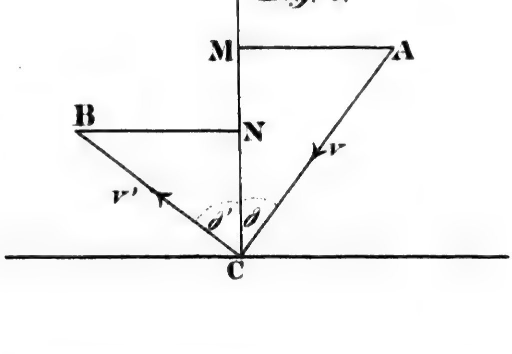
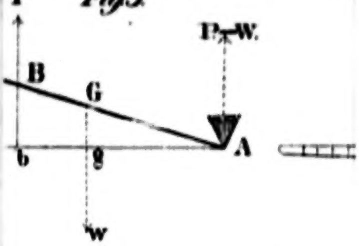


Fig. 6.



g. 7.

Fig. 3.



g. 7.

Fig. 8

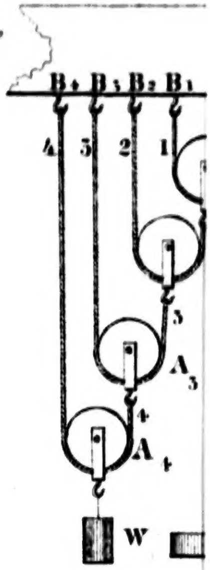
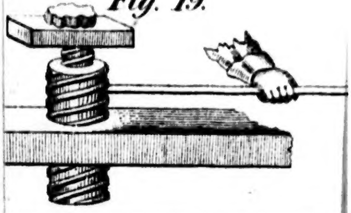


Fig. 19.



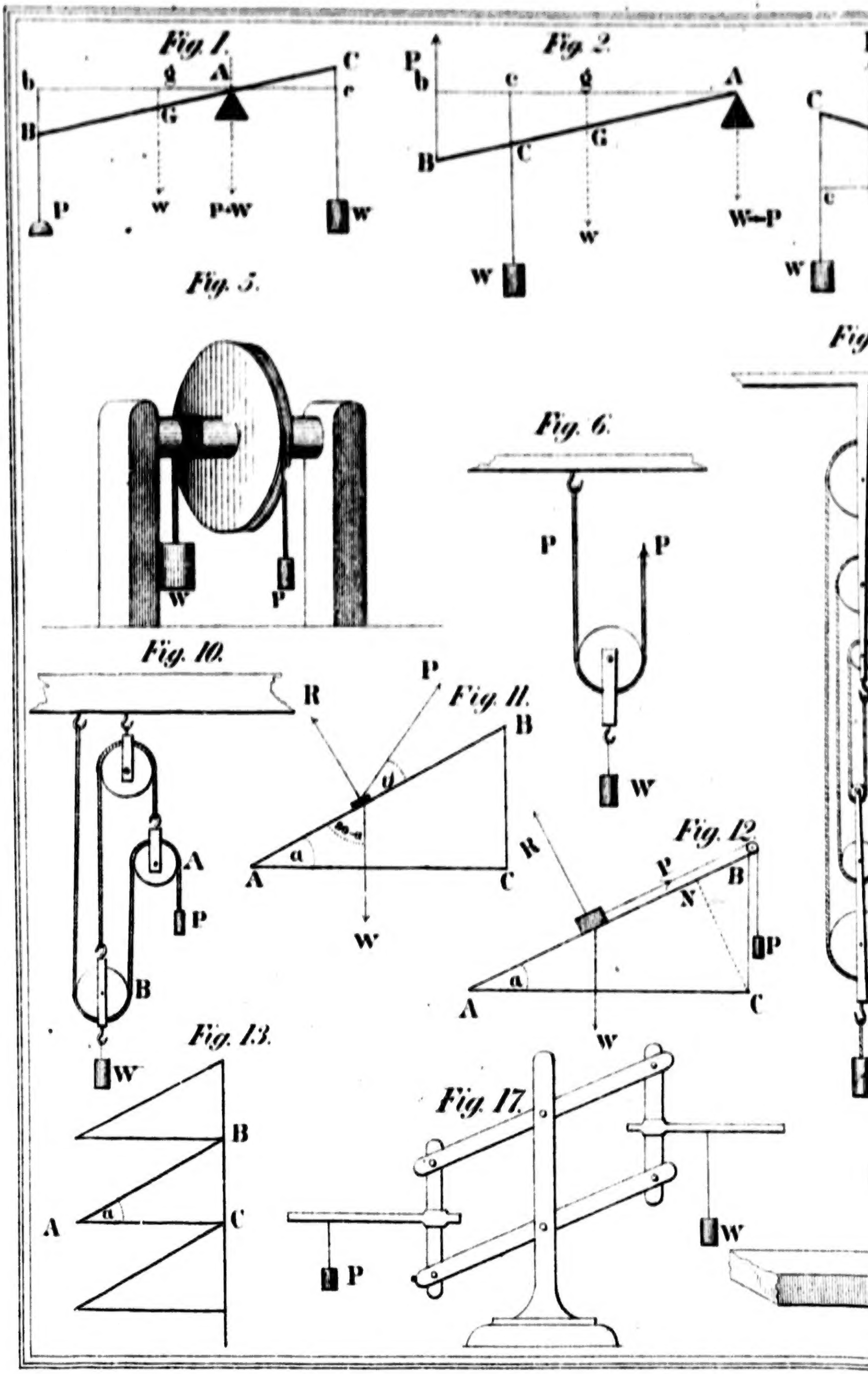


Fig. 1.
b g A C
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P w P+W W
Fig. 2.
P b e g A
B C G
W w W−P
Fig. 5.
W P
Fig. 6.
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W
Fig. 10.
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Fig. 11.
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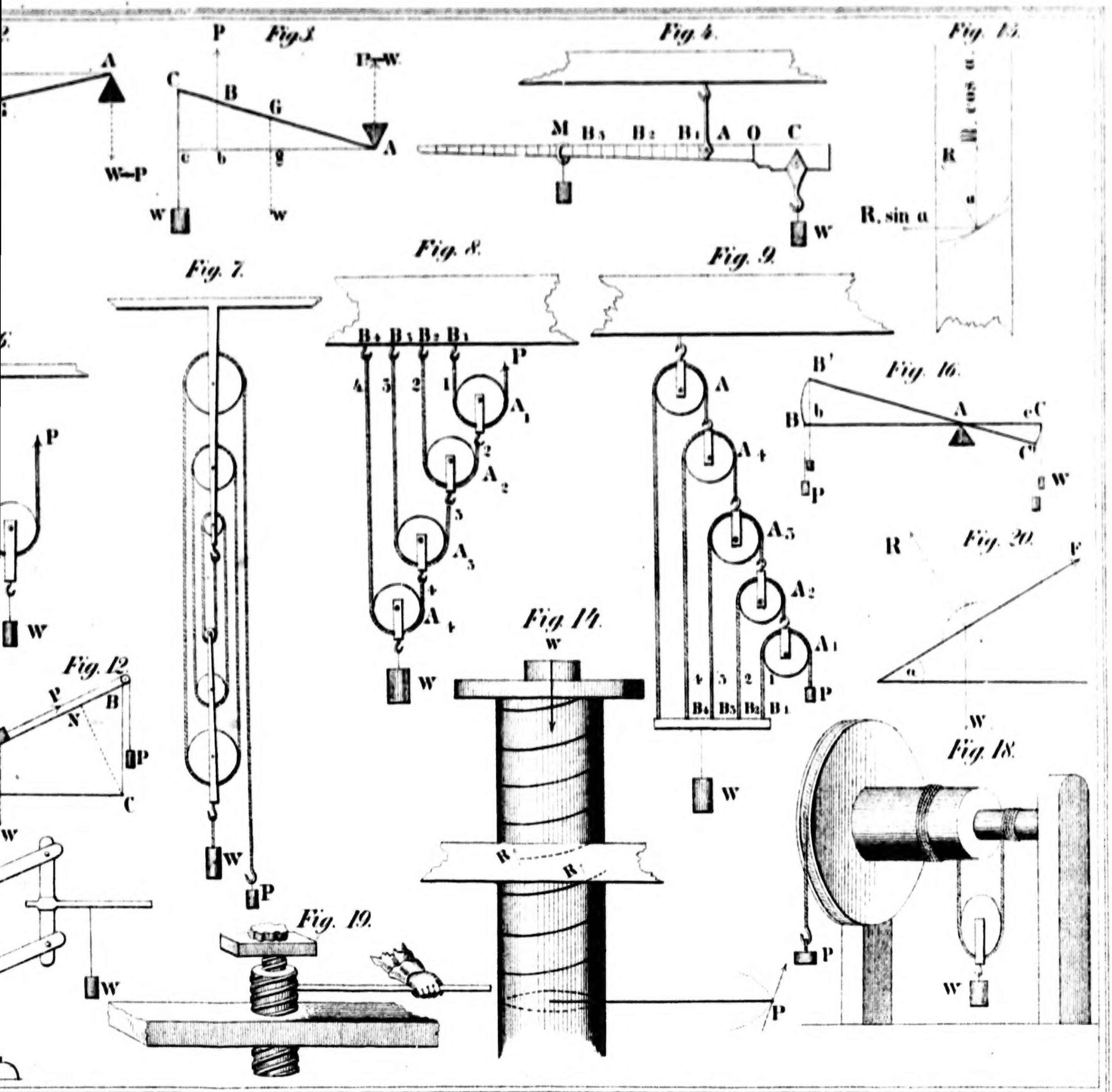

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